

Fine asymptotics for the consistent maximal displacement of branching Brownian motion

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Abstract

It is well-known that the maximal particle in a branching Brownian motion sits near $\sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$ at time t . One may then ask about the paths of particles near the frontier: how close can they stay to this critical curve? Two different approaches to this question have been developed. We improve upon the best-known bounds in each case, revealing new qualitative features including marked differences between the two approaches.

1 Introduction

A standard branching Brownian motion (BBM) begins with one particle at the origin. This particle moves as a Brownian motion, until an independent exponentially distributed time of parameter 1, at which point it is instantaneously replaced by two new particles. These particles independently repeat the stochastic behaviour of their parent relative to their start position, each moving like a Brownian motion and splitting into two at an independent exponentially distributed time of parameter 1.

Let $N(t)$ be the set of all particles alive at time t , and for a particle $v \in N(t)$ let $X_v(s)$ represent its position at time $s \leq t$ (or if v was not yet alive at time s , then the position of the unique ancestor of v that was alive at time s). If we define

$$M(t) = \max_{v \in N(t)} X_v(t)$$

then a simple law of large numbers shows that

$$\frac{M(t)}{t} \rightarrow \sqrt{2} \quad \text{as } t \rightarrow \infty.$$

One of the most striking results on BBM was given by Bramson [1], who calculated fine asymptotics for the distribution of $M(t)$, providing new results on travelling wave solutions to the FKPP equation; Hu and Shi [7] more recently (and for branching random walks rather than BBM) showed fluctuations in the almost-sure behaviour of $M(t)$ on this scale.

In summary, the *frontier* of the system — loosely speaking, the collection of particles near the maximum $M(t)$ at time t — is made up of particles that behave nothing like typical Brownian motions. In addition, the finer behaviour of the frontier is of interest as a tractable model that is conjectured — or in some cases proved — to belong to the same universality class as several important constructions arising in biology and statistical physics. It is natural, then, to ask what the paths of particles near $M(t)$ look like.

The problem of interest in this article is that of consistent maximal displacements: how close can particles stay to the critical line $\sqrt{2}u$, $u \geq 0$? There are (at least) two ways of making this question

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precise, each of which has been considered before for the related model of branching random walks. The first is to ask for which curves $f : [0, \infty) \rightarrow \mathbb{R}$ it is possible for particles to stay above $f(t)$ for all times $t \geq 0$. That is, when is

$$\nu(f) := \mathbb{P}(\forall t \geq 0, \exists v \in N(t) : X_v(u) > f(u) \quad \forall u \leq t)$$

non-zero? This was first considered by Jaffuel [8] (for branching random walks), who proved that there is a critical value $A_c = 3^{4/3}\pi^{2/3}2^{-7/6}$ such that if we set $f_a(t) = \sqrt{2t} - at^{1/3} - 1$ then $\nu(f_a) > 0$ if $a > A_c$, and $\nu(f_a) = 0$ if $a < A_c$.

The second approach is to look at maxima along recentered paths,

$$\lambda(v, t) = \sup_{s \in [0, t]} \{\sqrt{2}s - X_v(s)\},$$

and ask for the asymptotic behaviour of the minimum

$$\Lambda(t) = \min_{v \in N(t)} \lambda(v, t)$$

as $t \rightarrow \infty$. This quantity (or rather, again, its analogue for branching random walks) was studied by Fang and Zeitouni [2] and by Faraud, Hu and Shi [3], who showed that there is a critical value $a_c = 3^{1/3}\pi^{2/3}2^{-1/2}$ such that

$$\lim_{t \rightarrow \infty} \frac{\Lambda(t)}{t^{1/3}} = a_c.$$

To summarise, the two approaches to the question give similar results: in each case there appears to be a critical line on the $t^{1/3}$ scale above which particles cannot remain. We shall see, however, that if one peers more closely then the two situations are really quite different. Our first result is that not only is $\nu(f_{A_c}) > 0$ (which was previously unknown), but in fact particles may stay far above the curve f_{A_c} . Secondly, we are able to give asymptotics on the log scale for $\Lambda(t)$, for both the distribution function and the almost sure behaviour. These developments are redolent of the results of Bramson [1] and Hu and Shi [7]: although we are not able to gain quite such precise asymptotics, the results are very much of the same ilk. The proofs have certain elements in common with those found in [12], but are decidedly more involved, and we must develop several new techniques along the way.

We now state our three main theorems, which make precise the discussion above.

Theorem 1. *Let $A_c = 3^{4/3}\pi^{2/3}2^{-7/6}$. Define $g_\gamma : [0, \infty) \rightarrow \mathbb{R}$ by setting*

$$g_\gamma(t) = \sqrt{2t} - A_c t^{1/3} + t^\gamma - 1.$$

Then for any $\gamma < 1/3$,

$$\nu(g_\gamma) > 0.$$

Theorem 2. *Let $a_c = 3^{1/3}\pi^{2/3}2^{-1/2}$. Then for $b \in \mathbb{R}$,*

$$\frac{e^{O(\log \log t)}}{1 + t^{\sqrt{2}b+1/2}} \leq \mathbb{P}(\Lambda(t) - a_c t^{1/3} > b \log t) \leq t^{-\sqrt{2}b} e^{O(\log \log t)}.$$

Theorem 3. *$\Lambda(t)$ fluctuates on the logarithmic scale: for a_c as above,*

$$\liminf_{t \rightarrow \infty} \frac{\Lambda(t) - a_c t^{1/3}}{\log t} \in \left[-\frac{1}{2\sqrt{2}}, 0 \right]$$

almost surely, and

$$\limsup_{t \rightarrow \infty} \frac{\Lambda(t) - a_c t^{1/3}}{\log t} \in \left[\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$$

almost surely.

We should mention some limitations of these results. No upper bound is given in Theorem 1. We conjecture that, setting

$$h_{b,k}(t) = \sqrt{2}t - A_c t^{1/3} + b \frac{t^{1/3}}{\log^k(t+e)} - 1,$$

(where $\log^k t = (\log t)^k$) we should have $\nu(h_{b,k}) = 0$ for any $b > 0$ and $k > 0$. However we have no proof of this claim; the only rigorous upper bound we know of is that given by Jaffuel [8]. In fact, as mentioned above, Jaffuel considered branching random walks, but it is not difficult either to adapt his proof, or to apply his result together with standard tightness properties of Brownian motion, to achieve the same upper bound for BBM.

We conjecture that the lower bound given in Theorem 2 is accurate, and for Theorem 3 the correct values should be $-1/2\sqrt{2}$ and $1/2\sqrt{2}$ for the liminf and limsup respectively. Indeed, it seems from our proofs that the inaccuracy is introduced in giving the upper bound in Theorem 2.

The above results are stated only for standard BBM. There are however now well-known techniques (involving spines) for transferring the proofs to less straightforward cases, where for example each particle might give birth to a random number of new particles when it splits. In order to apply our methods we must only insist that the distribution of this random number has a finite second moment.

Further, we predict that our results should also hold for a wide class of branching random walks. Proving this, however, is far beyond our techniques, which rely on detailed estimates on the paths of Brownian motion not currently available in any generality for random walks.

1.1 Layout of the article

Our main tactic will be to develop detailed estimates for a single Brownian motion, and then to apply standard branching tools (the many-to-one and many-to-two lemmas) to deduce results for the branching system. Section 2 develops our main single-particle estimates, on the probability that a Brownian motion stays within a tube about a function f . This will then be used to prove Theorem 1 in Section 3. We move on to prove Theorem 2 in Section 4, which is then applied in Section 5 to prove Theorem 3.

1.2 Notation

Since we shall use several different probability measures in this article, to avoid confusion we do not use \mathbb{E} to denote expectation. For any probability measure P , we write $P(A)$ for the probability of the event A and $P[X]$ for the expectation of the random variable X .

2 A first single-particle estimate

In this section we are interested in estimating the probability that a Brownian motion stays close to a function f . We suppose that we have two twice continuously differentiable functions $f : [0, \infty) \rightarrow \mathbb{R}$ and $L : [0, \infty) \rightarrow (0, \infty)$, such that $f(0) = -x < 0$ and $f(0) + L(0) > 0$. Suppose that under \mathbb{P} , ξ_t , $t \geq 0$ is a Brownian motion started from 0, and let $\tilde{\xi}_t = \xi_t - f(t)$ and $K(t) = L(t)/2$. Write $(\mathcal{G}_t, t \geq 0)$ for the natural filtration of ξ_t . We begin with a simple application of Itô's formula.

Lemma 4.

$$\int_0^t \frac{K'(s)}{K(s)} \xi_s d\xi_s = \frac{K'(t)}{2K(t)} \xi_t^2 - \int_0^t \frac{\xi_s^2 K''(s)}{K(s)} ds + \int_0^t \frac{\xi_s^2 K'(s)^2}{K(s)^2} ds - \int_0^t \frac{K'(s)}{2K(s)} ds \quad (1)$$

and

$$\left[\xi, \int_0^\cdot \xi_s \frac{K'(s)}{K(s)} d\xi_s \right]_t = \int_0^t \xi_s \frac{K'(s)}{K(s)} ds \quad (2)$$

where $[X, Y]_t$ represents the quadratic covariation of X and Y at time t .

Proof. Itô's formula tells us that

$$\frac{\xi_t^2}{2} \frac{K'(t)}{K(t)} = \int_0^t \xi_s \frac{K'(s)}{K(s)} d\xi_s + \frac{1}{2} \int_0^t \xi_s^2 \frac{K''(s)}{K(s)} ds - \frac{1}{2} \int_0^t \xi_s^2 \frac{K'(s)^2}{K(s)^2} ds + \frac{1}{2} \int_0^t \frac{K'(s)}{K(s)} ds;$$

this gives (1), and (2) follows from the Kunita-Watanabe identity or by applying (1) together with the fact that $[X, Y]_t$ is the unique finite variation process such that $X_t Y_t - [X, Y]_t$ is a local martingale. \square

To begin with we will consider only $f \equiv -x$; we show that in this case, under reasonable conditions on L , we do not lose anything by considering only tubes that are symmetric about the origin.

Lemma 5. *Suppose that $|L'(0)|L(0) + |L'(t)|L(t) + \int_0^t L'(s)^2 ds \leq \bar{L}$ for all t . Then for $t \geq 0$ and any $0 < p < q < 1$,*

$$\begin{aligned} \mathbb{P}(x + \xi_s \in (0, L(s)) \quad \forall s \leq t, \quad x + \xi_t \in (pL(t), qL(t))) \\ \asymp_{\bar{L}} \mathbb{P}(|x - K(0) + \xi_s| < K(s) \quad \forall s \leq t, \quad x - K(0) + \xi_t \in ((2p-1)K(t), (2q-1)K(t))). \end{aligned}$$

Remark. Here the notation $g(t) \asymp_{\bar{L}} h(t)$ means that there exist constants $c(\bar{L}), C(\bar{L}) \in (0, \infty)$ depending only on \bar{L} such that $c(\bar{L})g(t) \leq h(t) \leq C(\bar{L})g(t)$ for all t in the given range.

Proof. Recall that \mathcal{G}_t is the natural filtration for the Brownian motion ξ_t and define a new measure \hat{P} by setting

$$\left. \frac{d\hat{P}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = e^{\int_0^t K'(s) d\xi_s - \frac{1}{2} \int_0^t K'(s)^2 ds}.$$

Then by Girsanov's theorem, under \hat{P} , $(\xi_t - K(t), t \geq 0)$ is a Brownian motion started from $K(0)$. But

$$K'(t)\xi_t = \int_0^t K'(s) d\xi_s + \int_0^t K''(s) \xi_s ds$$

so

$$e^{\int_0^t K'(s) d\xi_s - \frac{1}{2} \int_0^t K'(s)^2 ds} = e^{K'(t)\xi_t - \frac{1}{2} \int_0^t K'(s)^2 ds - \int_0^t K''(s) \xi_s ds}.$$

On the event $\{|x - K(0) + \xi_s| < K(s) \quad \forall s \leq t\}$, the exponent on the right-hand side is bounded above in modulus by \bar{L} . Thus

$$\begin{aligned} \mathbb{P}(x + \xi_s \in (0, L(s)) \quad \forall s \leq t, \quad x + \xi_t \in (pL(t), qL(t))) \\ = \mathbb{P}(|x - K(s) + \xi_s| < K(s) \quad \forall s \leq t, \quad x - K(t) + \xi_t \in ((2p-1)K(t), (2q-1)K(t))) \\ = \hat{P}(|x - K(0) + \xi_s| < K(s) \quad \forall s \leq t, \quad x - K(0) + \xi_t \in ((2p-1)K(t), (2q-1)K(t))) \\ = \mathbb{P} \left[e^{K'(t)\xi_t - \frac{1}{2} \int_0^t K'(s)^2 ds - \int_0^t K''(s) \xi_s ds} \mathbb{1}_{\{|x - K(0) + \xi_s| < K(s) \quad \forall s \leq t, \quad x - K(0) + \xi_t \in ((2p-1)K(t), (2q-1)K(t))\}} \right] \\ \asymp_{\bar{L}} \mathbb{P}(|x - K(0) + \xi_s| < K(s) \quad \forall s \leq t, \quad x - K(0) + \xi_t \in ((2p-1)K(t), (2q-1)K(t))) \end{aligned}$$

as claimed. \square

Our first real estimate tells us the probability that a Brownian motion stays within a tube of fixed width for a long time.

Lemma 6. *There exists a constant t_{tube} such that for all $t \geq t_{tube}$ and $x \in (-1, 1)$, $-1 \leq y < z \leq 1$,*

$$\mathbb{P}(|x + \xi_s| < 1 \quad \forall s \leq t, \quad x + \xi_t \in (y, z)) \asymp e^{-\pi^2 t/8} \cos\left(\frac{\pi x}{2}\right) \int_y^z \cos\left(\frac{\pi \nu}{2}\right) d\nu.$$

Proof. This is shown in [4, page 342]. For a proof more in keeping with the strategies in this article, one should consider the martingale

$$V_t = e^{\pi^2 t/8} \cos\left(\frac{\pi \xi_t}{2}\right) \mathbb{1}_{\{|x - \xi_s| < 1 \ \forall s \leq t\}}$$

and use it to define a change of measure, proceeding similarly to the proof of Lemma 5. \square

We now estimate the probability that $x + \xi_t$ stays within $(0, L(s))$ for all times $s \in [0, t]$. Define

$$\rho_L = \inf\{t > 0 : \int_0^t \frac{1}{L(s)^2} ds > t_{\text{tube}}\}.$$

Lemma 7. *Suppose that $|L'(0)|L(0) + |L'(t)|L(t) + \int_0^t L'(s)^2 ds \leq \bar{L}$ for all t . Then for $t \geq \rho_L$, for any $0 \leq p < q \leq 1$,*

$$\begin{aligned} \mathbb{P}(x + \xi_s \in (0, L(s)) \ \forall s \leq t, \ x + \xi_t \in (pL(t), qL(t))) \\ \asymp_{\bar{L}} e^{-\int_0^t \frac{\pi^2}{2L(s)^2} ds + \frac{1}{2} \log L(t) - \frac{1}{2} \log L(0)} \sin\left(\frac{\pi x}{L(0)}\right) \int_p^q \sin(\pi \nu) d\nu. \end{aligned}$$

Furthermore, for any $t > 0$,

$$\begin{aligned} \mathbb{P}(x + \xi_s \in (0, L(s)) \ \forall s \leq t, \ x + \xi_t \in (pL(t), qL(t))) \\ \lesssim_{\bar{L}} \left(x(q-p)q \frac{L(t)^2}{t^{3/2}}\right) \wedge \left((L(0) - x)(q-p)(1-p) \frac{L(t)^2}{t^{3/2}}\right) \wedge 1. \end{aligned}$$

Remark. Similarly to $\asymp_{\bar{L}}$, when we write $g(t) \lesssim_{\bar{L}} h(t)$ we mean that there exists a constant $C(\bar{L})$ depending only on \bar{L} such that $g(t) \leq C(\bar{L})h(t)$ for all t in the specified range.

Proof. We first note from Lemma 5 that

$$\begin{aligned} \mathbb{P}(x + \xi_s \in (0, L(s)) \ \forall s \leq t, \ x + \xi_t \in (pL(t), qL(t))) \\ \asymp_{\bar{L}} \mathbb{P}(|x - K(0) + \xi_s| < K(s) \ \forall s \leq t, \ x - K(0) + \xi_t \in ((2p-1)K(t), (2q-1)K(t))). \end{aligned}$$

We now adapt an idea from Novikov [10]. For $z \in (-1, 1)$ set

$$U_t = K(t)z + K(t) \int_0^t \frac{1}{K(s)} d\xi_s.$$

Then

$$dU_t = d\xi_t + U_t \frac{K'(t)}{K(t)} dt, \tag{3}$$

so if we define $\tilde{\mathbb{P}}$ by

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = e^{\int_0^t \xi_s \frac{K'(s)}{K(s)} d\xi_s - \frac{1}{2} \int_0^t \xi_s^2 \frac{K'(s)^2}{K(s)^2} ds}$$

then by Girsanov's theorem, $(\xi_t, t \geq 0)$ under $\tilde{\mathbb{P}}$ has the same distribution as $(U_t - U_0, t \geq 0)$ under \mathbb{P} . (Indeed, Girsanov's theorem together with (2) tells us that under $\tilde{\mathbb{P}}$

$$\xi_t - \int_0^t \xi_s \frac{K'(s)}{K(s)} ds$$

is a Brownian motion, whereas (3) tells us that under \mathbb{P}

$$U_t - \int_0^t U_s \frac{K'(s)}{K(s)} ds$$

is a Brownian motion. Thus the claim holds.) In particular

$$\begin{aligned} \mathbb{P}(|U_s| < K(s) \ \forall s \leq t, \ U_t \in ((2p-1)K(t), (2q-1)K(t))) \\ = \tilde{\mathbb{P}}(|K(0)z + \xi_s| < K(s) \ \forall s \leq t, \ K(0)z + \xi_t \in ((2p-1)K(t), (2q-1)K(t))). \end{aligned}$$

Thus, letting $z = x/K(0) - 1$,

$$\begin{aligned} \mathbb{P} \left[e^{\int_0^t \xi_s \frac{K'(s)}{K(s)} ds - \frac{1}{2} \int_0^t \xi_s^2 \frac{K'(s)^2}{K(s)^2} ds} \mathbb{1}_{\{|K(0)z + \xi_s| < K(s) \ \forall s \leq t, \ K(0)z + \xi_t \in ((2p-1)K(t), (2q-1)K(t))\}} \right] \\ = \tilde{\mathbb{P}}(|K(0)z + \xi_s| < K(s) \ \forall s \leq t, \ K(0)z + \xi_t \in ((2p-1)K(t), (2q-1)K(t))) \\ = \mathbb{P}(|U_s| < K(s) \ \forall s \leq t, \ U_t \in ((2p-1)K(t), (2q-1)K(t))) \\ = \mathbb{P} \left(\left| z + \int_0^s \frac{1}{K(r)} d\xi_r \right| < 1 \ \forall s \leq t, \ z + \int_0^t \frac{1}{K(r)} d\xi_r \in (2p-1, 2q-1) \right) \\ = \mathbb{P} \left(|z + \xi_s| < 1 \ \forall s \leq \int_0^t \frac{1}{K(r)^2} dr, \ z + \xi_{\int_0^t 1/K(r)^2 dr} \in (2p-1, 2q-1) \right). \end{aligned}$$

By (1) and the assumptions on L , since $\int_0^t K'(s)/K(s) ds = \log K(t) - \log K(0)$,

$$e^{\int_0^t \xi_s \frac{K'(s)}{K(s)} ds - \frac{1}{2} \int_0^t \xi_s^2 \frac{K'(s)^2}{K(s)^2} ds} \asymp_{\bar{L}} e^{\frac{1}{2} \log K(0) - \frac{1}{2} \log K(t)}$$

almost surely, so

$$\begin{aligned} \mathbb{P}(|K(0)z + \xi_s| < K(s) \ \forall s \leq t, \ K(0)z + \xi_t \in ((2p-1)K(t), (2q-1)K(t))) \\ \asymp_{\bar{L}} e^{\frac{1}{2} \log L(t) - \frac{1}{2} \log L(0)} \mathbb{P} \left(|z + \xi_s| < 1 \ \forall s \leq \int_0^t \frac{1}{K(r)^2} dr, \ z + \xi_{\int_0^t 1/K(r)^2 dr} \in (2p-1, 2q-1) \right). \end{aligned}$$

For $t \geq \rho_L$ the first part of the Lemma now follows from Lemma 6: for $t \geq \rho_L$,

$$\begin{aligned} \mathbb{P} \left(|z + \xi_s| < 1 \ \forall s \leq \int_0^t \frac{1}{K(r)^2} dr, \ z + \xi_{\int_0^t 1/K(r)^2 dr} \in (2p-1, 2q-1) \right) \\ \asymp e^{-\int_0^t \frac{\pi^2}{8K(s)^2} ds} \cos \left(\frac{\pi z}{2} \right) \int_{2p-1}^{2q-1} \cos \left(\frac{\pi \nu}{2} \right) d\nu \\ \asymp e^{-\int_0^t \frac{\pi^2}{2L(s)^2} ds} \sin \left(\frac{\pi x}{L(0)} \right) \int_p^q \sin(\pi \nu) d\nu, \end{aligned}$$

as required. For the second part, by Lemma 5 it suffices to show that for any $t > 0$,

$$\mathbb{P}(x + \xi_s \in (0, L(s)) \ \forall s \leq t, \ x + \xi_t \in (pL(t), qL(t))) \lesssim x(q-p)q \frac{L(t)^2}{t^{3/2}}.$$

But indeed,

$$\mathbb{P}(x + \xi_s \in (0, L(s)) \ \forall s \leq t, \ x + \xi_t \in (pL(t), qL(t))) \leq \mathbb{P}(x + \xi_s > 0 \ \forall s \leq t, \ x + \xi_t \in (pL(t), qL(t)))$$

and the result then follows by standard estimates on Bessel-3 processes (see for example the upper bound of [12, Lemma 3]). \square

Finally we apply Girsanov's theorem once more to consider tubes about the function f , rather than about 0.

Lemma 8. Suppose that $|L'(0)|L(0) + |L'(t)|L(t) + \int_0^t L'(s)^2 ds + \int_0^t |f''(s)|L(s) ds \leq \hat{L}$ for all t . If $f'(t) \geq 0$ then for $t \geq \rho_L$, for any $0 \leq p < q \leq 1$,

$$\begin{aligned} & e^{-\frac{1}{2} \int_0^t f'(s)^2 ds - \int_0^t \frac{\pi^2}{2L(s)^2} ds - f'(0)f(0) - qf'(t)L(t) + \frac{1}{2} \log L(t) - \frac{1}{2} \log L(0)} \sin\left(\frac{-\pi f(0)}{L(0)}\right) \int_p^q \sin(\pi\nu) d\nu \\ & \lesssim_{\hat{L}} \mathbb{P}(\xi_s - f(s) \in (0, L(s)) \quad \forall s \leq t, \quad \xi_t - f(t) \in (pL(t), qL(t))) \\ & \lesssim_{\hat{L}} e^{-\frac{1}{2} \int_0^t f'(s)^2 ds - \int_0^t \frac{\pi^2}{2L(s)^2} ds - f'(0)f(0) - pf'(t)L(t) + \frac{1}{2} \log L(t) - \frac{1}{2} \log L(0)} \sin\left(\frac{-\pi f(0)}{L(0)}\right) \int_p^q \sin(\pi\nu) d\nu. \end{aligned}$$

Furthermore, for any $t > 0$,

$$\begin{aligned} & \mathbb{P}(\xi_s - f(s) \in (0, L(s)) \quad \forall s \leq t, \quad \xi_t - f(t) \in (pL(t), qL(t))) \\ & \lesssim_{\hat{L}} e^{-\frac{1}{2} \int_0^t f'(s)^2 ds - f'(0)f(0) - pf'(t)L(t)} \\ & \quad \cdot \left(\left(-f(0)(q-p)q \frac{L(t)^2}{t^{3/2}} \right) \wedge \left((L(0) + f(0))(q-p)(1-p) \frac{L(t)^2}{t^{3/2}} \right) \wedge 1 \right). \end{aligned}$$

Proof. Define $\hat{\mathbb{P}}$ by setting

$$\left. \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = e^{\int_0^t f'(s) d\xi_s - \int_0^t f'(s)^2 ds}.$$

Note that, by integration by parts, since $\xi_0 = 0$,

$$\int_0^t f'(s) d\xi_s = f'(t)\xi_t - \int_0^t f''(s)\xi_s ds$$

so, applying Girsanov's theorem,

$$\begin{aligned} & \mathbb{P}(\xi_s - f(s) \in (0, L(s)) \quad \forall s \leq t, \quad \xi_t - f(t) \in (pL(t), qL(t))) \\ & = \hat{\mathbb{P}} \left[e^{-f'(t)\xi_t + \int_0^t f''(s)\xi_s ds + \frac{1}{2} \int_0^t f'(s)^2 ds} \mathbb{1}_{\{\xi_s - f(s) \in (0, L(s)) \quad \forall s \leq t, \quad \xi_t - f(t) \in (pL(t), qL(t))\}} \right] \\ & = \mathbb{P} \left[e^{-f'(t)(\xi_t - f(0)) - f'(t)f(t) + \int_0^t f''(s)(\xi_s - f(0)) ds + f'(t)f(t) - f'(0)f(0) - \int_0^t f'(s)^2 ds + \frac{1}{2} \int_0^t f'(s)^2 ds} \right. \\ & \quad \left. \cdot \mathbb{1}_{\{\xi_s - f(0) \in (0, L(s)) \quad \forall s \leq t, \quad \xi_t - f(0) \in (pL(t), qL(t))\}} \right] \\ & = \mathbb{P} \left[e^{-\frac{1}{2} \int_0^t f'(s)^2 ds - f'(0)f(0) - f'(t)(\xi_t - f(0)) + \int_0^t f''(s)(\xi_s - f(0)) ds} \right. \\ & \quad \left. \cdot \mathbb{1}_{\{\xi_s - f(0) \in (0, L(s)) \quad \forall s \leq t, \quad \xi_t - f(0) \in (pL(t), qL(t))\}} \right]. \end{aligned}$$

Now, on the event $\{\xi_s - f(0) \in (0, L(s)) \quad \forall s \leq t\}$,

$$\left| \int_0^t f''(s)(\xi_s - f(0)) ds \right| \leq \int_0^t |f''(s)|L(s) ds \leq \hat{L}$$

and on the event $\{\xi_t - f(0) \in (pL(t), qL(t))\}$, if $f'(t) \geq 0$ then $-f'(t)(\xi_t - f(0))$ is bounded below by $-qf'(t)L(t)$ and above by $-pf'(t)L(t)$. The result now follows from Lemma 7. \square

3 Proof of Theorem 1

We now apply the estimate obtained above to prove Theorem 1, which said that if we let $A_c = 3^{4/3}\pi^{2/3}2^{-7/6}$ and $\gamma \in [0, 1/3)$, then with positive probability there is always a particle above $\sqrt{2}t - A_c t^{1/3} + t^\gamma - 1$.

Proof of Theorem 1. Without loss of generality we may assume that γ is irrational. Choose J to be the unique integer in $(3\gamma/(1-3\gamma), 1+3\gamma/(1-3\gamma))$. Let

$$f(t) = \sqrt{2}t - A_c(t+1)^{1/3} + (t+1)^\gamma + \sum_{j=2}^J b_j(t+1)^{1/3-j(1/3-\gamma)} + \lambda \log(t+e) - C$$

and

$$L(t) = \alpha(t+1)^{1/3} + \beta(t+1)^\gamma$$

where C is any constant such that $f(0) < 0$ and $f(0) + L(0) > 0$. We will choose α, β, λ and b_2, \dots, b_J later.

It is clearly enough to prove that with probability bounded away from zero there is always a particle above $f(t)$. Define

$$\tilde{N}(t) = \#\{v \in N(t) : X_v(s) - f(s) \in (0, L(s)) \ \forall s \leq t, \ X_v(t) - f(t) \in (L(t) - 2, L(t) - 1)\}.$$

Note that, by Cauchy-Schwarz,

$$\mathbb{P}(\exists v \in N(t) : X_v(s) > f(s) \ \forall s \leq t) \geq \mathbb{P}(\tilde{N}(t) \geq 1) \geq \frac{\mathbb{P}[\tilde{N}(t)]^2}{\mathbb{P}[\tilde{N}(t)^2]}$$

so it suffices to show that $\mathbb{P}[\tilde{N}(t)]^2/\mathbb{P}[\tilde{N}(t)^2]$ is bounded below by a constant larger than zero.

We start by approximating some integrals using integration by parts.

$$\begin{aligned} \int_0^t \frac{1}{L(s)^2} ds &= \frac{1}{\alpha^2} \int_0^t \frac{1}{(s+1)^{2/3} (1 + \frac{\beta}{\alpha}(s+1)^{\gamma-1/3})^2} ds \\ &= \frac{3}{\alpha^2} \frac{(t+1)^{1/3}}{(1 + \frac{\beta}{\alpha}(t+1)^{\gamma-1/3})^2} - \frac{6\beta}{\alpha^3} \left(\frac{1}{3} - \gamma \right) \int_0^t \frac{(s+1)^{\gamma-1}}{(1 + \frac{\beta}{\alpha}(s+1)^{\gamma-1/3})^3} ds + O(1) \\ &= \frac{3}{\alpha^2} \frac{(t+1)^{1/3}}{(1 + \frac{\beta}{\alpha}(t+1)^{\gamma-1/3})^2} - \frac{6\beta(1/3-\gamma)}{\alpha^3} \frac{(t+1)^\gamma}{(1 + \frac{\beta}{\alpha}(t+1)^{\gamma-1/3})^3} \\ &\quad + \frac{18\beta^2}{\alpha^4\gamma} (1/3-\gamma)^2 \int_0^t \frac{(s+1)^{2\gamma-4/3}}{(1 + \frac{\beta}{\alpha}(s+1)^{\gamma-1/3})^4} ds + O(1). \end{aligned}$$

We can continue in this way; integration by parts tells us that

$$\begin{aligned} \int_0^t \frac{(s+1)^{k\gamma-(k+2)/3}}{(1 + \frac{\beta}{\alpha}(s+1)^{\gamma-1/3})^{k+2}} ds &= \frac{1}{k\gamma - (k-1)/3} \frac{(t+1)^{k\gamma-(k-1)/3}}{(1 + \frac{\beta}{\alpha}(t+1)^{\gamma-1/3})^{k+3}} \\ &\quad - \frac{(k+2)\beta(1/3-\gamma)}{\alpha(k\gamma - (k-1)/3)} \int_0^t \frac{(s+1)^{(k+1)\gamma-(k+3)/3}}{(1 + \frac{\beta}{\alpha}(s+1)^{\gamma-1/3})^{k+3}} ds + O(1) \end{aligned}$$

so (by our choice of J) there exist constants $\Gamma_0, \dots, \Gamma_J$ such that

$$\int_0^t \frac{1}{L(s)^2} ds = \sum_{j=0}^J \Gamma_j \frac{(t+1)^{j\gamma-(j-1)/3}}{(1 + \frac{\beta}{\alpha}(t+1)^{\gamma-1/3})^{j+3}} + O(1).$$

Thus in fact there exist constants $\tilde{\Gamma}_0, \dots, \tilde{\Gamma}_J$ such that

$$\int_0^t \frac{1}{L(s)^2} ds = \sum_{j=0}^J \tilde{\Gamma}_j (t+1)^{j\gamma-(j-1)/3} + O(1),$$

and we see from above that in particular $\tilde{\Gamma}_0 = 3/\alpha^2$ and $\tilde{\Gamma}_1 = -2\beta/\alpha^3\gamma$.

Also,

$$\frac{1}{2} \int_0^t f'(s)^2 ds = t - \sqrt{2} A_c (t+1)^{1/3} + \sqrt{2} (t+1)^\gamma + \sum_{j=2}^J \sqrt{2} b_j (t+1)^{j\gamma-(j-1)/3} + \sqrt{2} \lambda \log(t+e) + O(1)$$

and one may check that

$$\int_0^t |f''(s)| L(s) ds = O(1), \quad L'(t) L(t) = O(1), \quad \int_0^t L'(s)^2 ds = O(1) \quad \text{and} \quad \int_0^t L''(s) L(s) ds = O(1)$$

so that the conditions of Lemma 8 hold for some constant \hat{L} .

We now choose $\alpha = 3^{1/3} \pi^{2/3} 2^{-1/6}$ so that $-\sqrt{2} A_c + 3\pi^2/2\alpha^2 = -\sqrt{2}\alpha$, then $\beta = 3\gamma/(1-3\gamma)$ so that $\sqrt{2} - \pi^2\beta/\alpha^3\gamma = -\sqrt{2}\beta$. Finally for $j \geq 2$ we choose $b_j = \pi^2 \tilde{\Gamma}_j / 2\sqrt{2}$. Then

$$\frac{1}{2} \int_0^t f'(s)^2 ds + \int_0^t \frac{\pi^2}{2L(s)^2} ds = t - \sqrt{2}\alpha(t+1)^{1/3} - \sqrt{2}\beta(t+1)^\gamma + \sqrt{2}\lambda \log(t+e) + O(1).$$

Further, on $\{\xi_t - f(t) \in (L(t) - 2, L(t) - 1)\}$,

$$f'(t) L(t) = \sqrt{2}\alpha(t+1)^{1/3} + \sqrt{2}\beta(t+1)^\gamma + O(1),$$

$$\frac{1}{2} \log L(t) - \frac{1}{2} \log L(0) = \frac{1}{6} \log(t+1) + O(1),$$

and

$$\int_{1-2/L(t)}^{1-1/L(t)} \sin(\pi\nu) d\nu = e^{O(1)} L(t)^{-2} = e^{-\frac{2}{3} \log(t+1) + O(1)}.$$

Thus, applying the many-to-one lemma (see for example [5]) and Lemma 8,

$$\begin{aligned} \mathbb{P}[\tilde{N}(t)] &= e^t \mathbb{P}(\xi_s - f(s) \in (0, L(s)) \quad \forall s \leq t, \quad \xi_t - f(t) \in (L(t) - 2, L(t) - 1)) \\ &\asymp e^{\sqrt{2}\alpha(t+1)^{1/3} + \sqrt{2}\beta(t+1)^\gamma - \sqrt{2}\lambda(t+e) - \sqrt{2}\alpha(t+1)^{1/3} - \sqrt{2}\beta(t+1)^\gamma + \frac{1}{6} \log(t+1) - \frac{2}{3} \log(t+1)} \\ &\asymp t^{-\sqrt{2}\lambda - 1/2}. \end{aligned}$$

We now check the second moment of $\tilde{N}(t)$. We apply the many-to-few lemma (again, see [5]). Suppose that T is an independent exponentially distributed random variable of parameter 2 and, given T , $(\xi_s^{(1)}, s \geq 0)$ and $(\xi_s^{(2)}, s \geq 0)$ are standard Brownian motions such that

- $\xi_s^{(1)} = \xi_s^{(2)}$ for all $s \in [0, T]$;
- $(\xi_{T+s}^{(1)} - \xi_T^{(1)}, s \geq 0)$ and $(\xi_{T+s}^{(2)} - \xi_T^{(2)}, s \geq 0)$ are independent given \mathcal{G}_T .

For $i = 1, 2$, let

$$A_t^{(i)} = \{\xi_s^{(i)} - f(s) \in (0, L(s)) \quad \forall s \leq t\}$$

and

$$C_t^{(i)} = \{\xi_t^{(i)} - f(t) \in (L(t) - 2, L(t) - 1)\}$$

and define

$$\Theta_t = A_t^{(1)} \cap A_t^{(2)} \cap C_t^{(1)} \cap C_t^{(2)}.$$

Then the many-to-few lemma tells us that

$$\mathbb{P}[\tilde{N}(t)^2] = \mathbb{P}[\tilde{N}(t)] + 2 \int_0^t e^{2t-s} \mathbb{P}(\Theta_t | T = s) ds.$$

Recall the definition of t_{tube} from Lemma 6, and define

$$\bar{\rho}_L(t) = \sup \left\{ s < t : \int_s^t \frac{1}{L(u)^2} du > t_{\text{tube}} \right\}.$$

Applying the Markov property and then Lemma 8, for any $s \leq \bar{\rho}_L(t)$ and $j = 0, \dots, \lfloor L(s) \rfloor$,

$$\begin{aligned}
& \mathbb{P} \left(\Theta_t | A_T^{(1)} \cap A_T^{(2)} \cap \{ \xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j] \} \cap \{ T = s \} \right) \\
& \leq \sup_{z \in (j, j+1]} \mathbb{P}(\xi_u - f(s+u) + f(s) + L(s) - z \in (0, L(s+u)) \quad \forall u \leq t-s, \\
& \quad \xi_{t-s} - f(t) + f(s) + L(s) - z \in (L(t) - 2, L(t) - 1))^2 \\
& \asymp \sup_{z \in (j, j+1]} e^{-\int_s^t f'(u)^2 du - \int_s^t \frac{\pi^2}{L(u)^2} du - 2f'(s)(z-L(s)) - 2f'(t)L(t) + \log L(t) - \log L(s)} \\
& \quad \cdot \sin^2 \left(\frac{\pi z}{L(s)} \right) \left(\int_{1-2/L(t)}^{1-1/L(t)} \sin(\pi \nu) d\nu \right)^2 \\
& \asymp e^{-2(t-s) + 2\sqrt{2}\alpha(t+1)^{1/3} - 2\sqrt{2}\alpha(s+1)^{1/3} + 2\sqrt{2}\beta(t+1)^\gamma - 2\sqrt{2}\beta(s+1)^\gamma - 2\sqrt{2}\lambda(t+e) + 2\sqrt{2}\lambda \log(s+e)} \\
& \quad \cdot e^{-2\sqrt{2}j + 2\sqrt{2}\alpha(s+1)^{1/3} + 2\sqrt{2}\beta(s+1)^\gamma - 2\sqrt{2}\alpha(t+1)^{1/3} - 2\sqrt{2}\beta(t+1)^\gamma + \frac{1}{3} \log(t+1) - \frac{1}{3} \log(s+1)} \\
& \quad \cdot \sin^2 \left(\frac{\pi z}{L(s)} \right) \left(\int_{1-2/L(t)}^{1-1/L(t)} \sin(\pi \nu) d\nu \right)^2 \\
& \lesssim (j+1)^2 e^{-2\sqrt{2}j - 2(t-s)} (t+1)^{-2\sqrt{2}\lambda-1} (s+1)^{2\sqrt{2}\lambda-1}.
\end{aligned}$$

Also, for $s \geq \rho_L$,

$$\begin{aligned}
& \mathbb{P}(A_T^{(1)} \cap \{ \xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j] \} | T = s) \\
& \asymp e^{-\frac{1}{2} \int_0^s f'(u)^2 du - \int_0^s \frac{\pi^2}{2L(u)^2} du - f'(0)f(0) - f'(s)(L(s) - j) + \frac{1}{2} \log L(s) - \frac{1}{2} \log L(0)} \\
& \quad \cdot \sin \left(\frac{\pi f(0)}{L(0)} \right) \int_{1-(j+1)/L(s)}^{1-j/L(s)} \sin(\pi \nu) d\nu \\
& \asymp e^{-s + \sqrt{2}\alpha(s+1)^{1/3} + \sqrt{2}\beta(s+1)^\gamma - \sqrt{2}\lambda \log(s+e) + \sqrt{2}j - \sqrt{2}\alpha(s+1)^{1/3} - \sqrt{2}\beta(s+1)^\gamma + \frac{1}{6} \log(s+1)} \\
& \quad \cdot \sin \left(\frac{\pi f(0)}{L(0)} \right) \int_{1-(j+1)/L(s)}^{1-j/L(s)} \sin(\pi \nu) d\nu \\
& \lesssim (j+1) e^{\sqrt{2}j-s} (s+1)^{-1/2-\sqrt{2}\lambda}.
\end{aligned}$$

Thus for $s \in [\rho_L, \bar{\rho}_L(t)]$

$$\begin{aligned}
& \mathbb{P}(\Theta_t | T = s) \\
& \lesssim \sum_{j=0}^{\lfloor L(s) \rfloor} (j+1)^3 e^{-\sqrt{2}j-2t+s} (t+1)^{-2\sqrt{2}\lambda-1} (s+1)^{\sqrt{2}\lambda-3/2} \\
& \asymp e^{-2t+s} (t+1)^{-2\sqrt{2}\lambda-1} (s+1)^{\sqrt{2}\lambda-3/2}.
\end{aligned}$$

and hence, provided $\sqrt{2}\lambda - 3/2 < -1$,

$$\int_{\rho_L}^{\bar{\rho}_L(t)} e^{2t-s} \mathbb{P}(\Theta_t | T = s) ds \lesssim (t+1)^{-2\sqrt{2}\lambda-1}.$$

Similarly,

$$\begin{aligned}
& \int_0^{\rho_L} e^{2t-s} \mathbb{P}(\Theta_t | T = s) ds \\
& \leq \int_0^{\rho_L} e^{2t} \sum_{j=0}^{\lfloor L(s) \rfloor} \mathbb{P} \left(\Theta_t \mid A_T^{(1)} \cap A_T^{(2)} \cap \{\xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j]\} \cap \{T = s\} \right) ds \\
& \lesssim \int_0^{\rho_L} e^{2t} \sum_{j=0}^{\lfloor L(s) \rfloor} e^{-2\sqrt{2}j - 2(t-s) - 2\sqrt{2}\lambda(\log(t+e) - \log(s+e)) - \frac{1}{3}\log(s+e) + \frac{1}{3}\log(t+e)} \frac{(j+1)^2}{(s+1)^{2/3}} \frac{1}{(t+1)^{4/3}} ds \\
& \asymp (t+1)^{-2\sqrt{2}\lambda-1}.
\end{aligned}$$

Finally we consider the integral from $\bar{\rho}_L(t)$ to t . Note that $t - \bar{\rho}_L(t) \asymp (t+1)^{2/3}$. For any $s \in [\bar{\rho}_L(t), t-1)$, by the last part of Lemma 8,

$$\begin{aligned}
& \mathbb{P} \left(\Theta_t \mid A_T^{(1)} \cap A_T^{(2)} \cap \{\xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j]\} \cap \{T = s\} \right) \\
& \leq \sup_{z \in (j, j+1]} \mathbb{P}(\xi_u - f(s+u) + f(s) + L(s) - z \in (0, L(s+u)) \quad \forall u \leq t-s, \\
& \quad \xi_{t-s} - f(t) + f(s) + L(s) - z \in (L(t) - 2, L(t) - 1))^2 \\
& \lesssim \sup_{z \in (j, j+1]} \left(e^{-\frac{1}{2} \int_s^t f'(u)^2 du - f'(s)z + f'(s)L(s) - f'(t)L(t)} \frac{z}{L(t)} \frac{2}{L(t)} \frac{L(t)^2}{(t-s)^{3/2}} \right)^2 \\
& \lesssim (j+1)^2 e^{-2\sqrt{2}j - 2(t-s)} (t-s)^{-3}.
\end{aligned}$$

Thus, for $s \in [\bar{\rho}_L(t), t-1)$,

$$\begin{aligned}
\mathbb{P}(\Theta_t | T = s) & \lesssim \sum_{j=0}^{\lfloor L(s) \rfloor} (j+1)^3 e^{-\sqrt{2}j - 2t+s} (t-s)^{-3} (s+1)^{-\sqrt{2}\lambda-1/2} \\
& \asymp e^{-2t+s} (t-s)^{-3} (t+1)^{-\sqrt{2}\lambda-1/2}.
\end{aligned}$$

Similarly for $s \in [t-1, t)$,

$$\mathbb{P}(\Theta_t | T = s) \lesssim e^{-2t+s} (t+1)^{-\sqrt{2}\lambda-1/2}.$$

Hence

$$\begin{aligned}
& \int_{\bar{\rho}_L(t)}^t e^{2t-s} \mathbb{P}(A_t^{(1)} \cap A_t^{(2)} \cap C_t^{(1)} \cap C_t^{(2)} | T = s) ds \\
& \lesssim (t+1)^{-\sqrt{2}\lambda-1/2} \left(\int_{\bar{\rho}_L(t)}^{t-1} (t-s)^{-3} ds + \int_{t-1}^t 1 ds \right) \\
& \asymp (t+1)^{-\sqrt{2}\lambda-1/2}.
\end{aligned}$$

Putting all of this together, we obtain

$$\mathbb{P}[\tilde{N}(t)^2] \lesssim \mathbb{P}[\tilde{N}(t)] + (t+1)^{-2\sqrt{2}\lambda-1} + (t+1)^{-\sqrt{2}\lambda-1/2}.$$

Recalling from earlier that

$$\mathbb{P}[\tilde{N}(t)] \asymp (t+1)^{-\sqrt{2}\lambda-1/2}$$

and choosing $\lambda = -1/2\sqrt{2}$ we see that

$$\mathbb{P}(\tilde{N}(t) \geq 1) \geq \frac{\mathbb{P}[\tilde{N}(t)]^2}{\mathbb{P}[\tilde{N}(t)^2]} \geq c_0 > 0$$

for all large t . □

4 Proof of Theorem 2

For $x = x_t > 0$, we now want to estimate

$$\mathbb{P}_x(\exists v \in N(t) : X_v(u) > -x + \sqrt{2}u \quad \forall u \leq t).$$

The general tactic is to consider instead the set of particles that remain between $-x + \sqrt{2}u$ and $-x + \sqrt{2}u + L_t(u)$ for all times $u \leq t$. The reason for this restriction is that with no upper boundary imposed, events of vanishing probability have a distorting effect on the moments of the number of particles with large position. To this end, for each $t > 1$ we shall now fix twice continuously differentiable functions $f_t : [0, t] \rightarrow \mathbb{R}$, $g_t : [0, t] \rightarrow \mathbb{R}$, and $L_t : [0, t] \rightarrow (0, \infty)$. To simplify notation we shall often drop the subscript and write f , g and L where we mean f_t , g_t and L_t .

We choose

$$L(s) = L_t(s) = a(t + 2 - s)^{1/3}$$

where $a = a_c = 3^{1/3}\pi^{2/3}2^{-1/2}$. We also let $b' = 1 - 0 \wedge b$ and $\mu = \sqrt{2} - b' \frac{\log t}{t}$ and define

$$f(s) = f_t(s) = -a(t + e)^{1/3} + (b + b') \log t + \mu s$$

and

$$g(s) = g_t(s) = -a(t + e)^{1/3} + b \log t + \sqrt{2}s.$$

(The idea is that we would like to work with g_t , since we want $x_t = a(t + e)^{1/3} + b \log t$, but if $b < 0$ then $g_t(0) + L_t(0) < 0$; so we work first with f_t in order to develop a way around the problem.) We note that

$$\int_0^s \frac{\pi^2}{2L(u)^2} du = \sqrt{2}L(0) - \sqrt{2}L(s), \quad (4)$$

while

$$\frac{1}{2} \int_0^s f'(u)^2 du = \frac{1}{2} \mu^2 s = s - (\sqrt{2}b' \log t) \frac{s}{t} + O(1) \quad (5)$$

and

$$f'(0)f(0) = \mu f(0) = -\sqrt{2}at^{1/3} + \sqrt{2}(b + b') \log t + O(1). \quad (6)$$

It is also easy to check that the conditions of Lemma 8 hold, giving us the following immediate application.

Lemma 9. *For $s \in [\rho_L, t]$, for the choice of f and L given above and $0 \leq p < q \leq 1$,*

$$\begin{aligned} & ((b + b') \log t) e^{-s + (1-q)\sqrt{2}a(t+e-s)^{1/3} - (\sqrt{2}b+1/2) \log t + (\sqrt{2}b' \log t)(s/t-1) + \frac{1}{6} \log(t+e-s)} \int_p^q \sin(\pi\nu) d\nu \\ & \lesssim \mathbb{P}(\xi_u - f(u) \in (0, L(u)) \quad \forall u \leq s, \quad \xi_s - f(s) \in (pL(s), qL(s))) \\ & \lesssim ((b + b') \log t) e^{-s + (1-p)\sqrt{2}a(t+e-s)^{1/3} - (\sqrt{2}b+1/2) \log t + (\sqrt{2}b' \log t)(s/t-1) + \frac{1}{6} \log(t+e-s)} \int_p^q \sin(\pi\nu) d\nu. \end{aligned}$$

For any $s > 0$,

$$\begin{aligned} & \mathbb{P}(\xi_u - f(u) \in (0, L(u)) \quad \forall u \leq s, \quad \xi_s - f(s) \in (pL(s), qL(s))) \\ & \lesssim \left(\left(((b + b') \log t)(q - p)(1 - p) \frac{(t + e - s)^{2/3}}{s^{3/2}} \right) \wedge 1 \right) \\ & \quad \cdot e^{-s + \sqrt{2}at^{1/3} - \sqrt{2}pa(t+e-s)^{1/3} - \sqrt{2}b \log t + (\sqrt{2}b' \log t)(s/t-1)}. \end{aligned}$$

4.1 A martingale and its change of measure

Suppose that $h : [0, \infty) \rightarrow \mathbb{R}$ is linear, i.e. $h(s) = As + B$ for some constants A and B . Define

$$\zeta^{h,L}(s) = e^{h'(s)\xi_s - \frac{1}{2} \int_0^s h'(u)^2 du + \int_0^s \frac{\pi^2}{2L(u)^2} du + \frac{L'(s)}{2L(s)} (\xi_s - h(s))^2 - \int_0^t \frac{L''(u)}{2L(u)} (\xi_u - h(u))^2 du - \frac{1}{2} \log L(s)} \cdot \sin\left(\frac{\pi(\xi_s - h(s))}{L(s)}\right) \mathbb{1}_{\{\xi_u - h(u) \in (0, L(u)) \ \forall u \leq s\}}.$$

Provided that L is twice continuously differentiable, the process $(\zeta^{h,L}(s), s \in [0, t])$ is a martingale. The proof of this fact simply involves applying Itô's formula; see [6] for details. We use it to define a new measure $\mathbb{Q}^{h,L}$ by setting

$$\left. \frac{d\mathbb{Q}^{h,L}}{d\mathbb{P}} \right|_{\mathcal{G}_s} = \frac{\zeta^{h,L}(s)}{\zeta^{h,L}(0)}, \quad s \in [0, t].$$

Note that

$$\mathbb{Q}^{h,L}(\xi_s \in (0, L(s)) \ \forall s \leq t) = \mathbb{P}\left[\frac{\zeta^{h,L}(t)}{\zeta^{h,L}(0)} \mathbb{1}_{\{\xi_s - h(s) \in (0, L(s)) \ \forall s \leq t\}}\right] = \mathbb{P}\left[\frac{\zeta^{h,L}(t)}{\zeta^{h,L}(0)}\right] = 1.$$

Also note that, for the choice of f and L above,

$$\frac{\zeta^{f,L}(s)}{\zeta^{f,L}(0)} = e^{\mu\xi_s - \frac{1}{2}\mu^2 s + \int_0^s \frac{\pi^2}{2L(u)^2} du + \frac{1}{2} \log L(0) - \frac{1}{2} \log L(s) + O(1)} \frac{\sin\left(\frac{\pi(\xi_s - f(s))}{L(s)}\right)}{\sin\left(\frac{-\pi f(0)}{L(0)}\right)} \mathbb{1}_{\{\xi_u - f(u) \in (0, L(u)) \ \forall u \leq s\}}.$$

4.2 Stopping lines

We want to bound from above the number of particles staying above a fixed line. In order to stay above $f(s)$, say, for all $s \leq t$, a particle has two options: it can stay within $(f(s), f(s) + L(s))$ or it can venture above $f(s) + L(s)$ at some time s . We can easily bound the number of particles in the former scenario using the estimates developed in the previous section. However we also need to bound the number of particles in the second scenario. Particles that do venture above $f(s) + L(s)$ might have many descendants at time t that have stayed above f , so instead of counting all of these descendants at time t , which might give misleading moment estimates, we want to stop particles as soon as they hit (or rather get close to) the upper line. This is essentially the definition of a *stopping line*: for a rigorous definition, for a particle v let σ_v be its time of birth and τ_v its time of death. For a function $h : [0, t] \rightarrow \mathbb{R}$ let $U_v = \inf\{s \in [0, t] : X_v(s) = h(s)\}$, with the convention that $\inf \emptyset = \infty$. Recall that $X_v(s)$ is defined not only for $s \in [\sigma_v, \tau_v)$ but for all $s \in [0, \tau_v)$ by considering the position of the unique ancestor of v that was alive at time s . Let

$$N_U(t) = \left\{ v \in \bigcup_{s \leq t} N(s) : \sigma_v \leq U_v < \tau_v \right\}.$$

The many-to-one lemma for stopping lines [9] tells us that for any measurable function $\kappa : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$, if we set $U = \inf\{s \in [0, t] : \xi_s = h(s)\}$ then

$$\mathbb{P}\left[\sum_{v \in N_U(t)} \kappa(X_v(U_v), U_v)\right] = \mathbb{P}[e^U \kappa(\xi_U, U) \mathbb{1}_{\{U \leq t\}}].$$

4.3 The upper bound in Theorem 2

We want to calculate the probability that a particle stays above $g(s)$ for all $s \leq t$. We split this event up into subevents. First, a particle might go above $f(s) + L(s)$ for some s ; or it might venture below $f(s)$ for some s . To be more precise, for a particle $v \in N(t)$ define

$$S_v = \inf\{s \in [0, t] : X_v(s) \geq f_t(s) + L_t(s) - 1\}$$

and

$$T_v = \inf\{s \in [0, t] : X_v(s) < f_t(s) + 1\}.$$

We also write $S = \inf\{s \in [0, t] : \xi_s \geq f_t(s) + L_t(s) - 1\}$ and $T = \inf\{s \in [0, t] : \xi_s < f_t(s) + 1\}$.

Lemma 10.

$$\mathbb{P}[\#\{v \in N_S(t) : S_v < T_v\}] \leq t^{-\sqrt{2}b} e^{O(\log \log t)}.$$

Proof. By the many-to-one lemma for stopping lines,

$$\begin{aligned} \mathbb{P}[\#\{v \in N_S(t) : S_v < T_v\}] &= \mathbb{P}[e^S \mathbb{1}_{\{S < T\}}] = \mathbb{Q}^{f, L} \left[\frac{e^S}{\zeta^{f, L}(S)} \mathbb{1}_{\{S < T\}} \right] \\ &= \mathbb{Q}^{f, L} \left[e^{S - \mu \xi_S + \frac{1}{2} \mu^2 S - \int_0^S \frac{\pi^2}{2L(u)^2} du + \frac{1}{2} \log L(S) - \frac{1}{2} \log L(0) + O(1)} \frac{\sin\left(\frac{-\pi f(0)}{L(0)}\right)}{\sin\left(\frac{-\pi(\xi_S - f(S))}{L(S)}\right)} \mathbb{1}_{\{S < T\}} \right]. \end{aligned}$$

Now note that $\xi_S = \mu S + f(0) + L(S) - 1$, so (also using (4), (5), (6))

$$\begin{aligned} \mathbb{P}[\#\{v \in N(S) : S_v < T_v\}] &= \mathbb{Q}^{f, L} \left[e^{S - \frac{1}{2} \mu^2 S - \mu f(0) - \mu L(S) + \sqrt{2}L(S) - \sqrt{2}L(0) + \frac{3}{2} \log L(S) - \frac{3}{2} \log L(0) + O(\log \log t)} \mathbb{1}_{\{S < T\}} \right] \\ &= \mathbb{Q}^{f, L} \left[e^{(\sqrt{2}b' \log t)S/t + \sqrt{2}L(0) - \sqrt{2}(b+b') \log t - \sqrt{2}L(0) + \frac{3}{2} \log L(S) - \frac{3}{2} \log L(0) + O(\log \log t)} \mathbb{1}_{\{S < T\}} \right] \\ &\leq t^{-\sqrt{2}b} e^{O(\log \log t)}. \end{aligned} \quad \square$$

Remark. In order to get our upper and lower bounds to agree, we would like to be able to say that (for a suitable choice of f and L , not necessarily those above)

$$\mathbb{Q} \left[e^{\frac{3}{2} \log L(S) - \frac{3}{2} \log L(0)} \mathbb{1}_{\{S < T\}} \right] \leq t^{-1/2} e^{o(\log t)}.$$

We do not know how to achieve this; in fact the best upper bound we know here is the trivial one, 1, which we use above.

We now consider those particles v for which $T_v < S_v$. Let

$$\tilde{S}_v = \inf\{s \in (T_v, t] : X_v(s) - g(s) > L(s) - 2\}$$

and

$$\tilde{T}_v = \inf\{s \in (T_v, t] : X_v(s) - g(s) \leq 0\}.$$

For $w \in N_T(t)$, define

$$N_{\tilde{S}}^w(t) = \{v \in N_{\tilde{S}}(t) : w \leq v\},$$

the set of descendants of w (possibly including w itself) that are in $N_{\tilde{S}}(t)$. Let \mathcal{F}_T be the sigma-field containing all information about each particle w up until time T_w .

Lemma 11.

$$\mathbb{P}[\#\{v \in N_{\tilde{S}}(t) : T_v < S_v, \tilde{S}_v < \tilde{T}_v\}] \leq t^{-\sqrt{2}b} e^{O(\log \log t)}.$$

Proof. By the tower law,

$$\mathbb{P}[\#\{v \in N_{\tilde{S}}(t) : T_v < S_v, \tilde{S}_v < \tilde{T}_v\}] = \mathbb{P} \left[\sum_{w \in N_T(t)} \mathbb{P} \left[\sum_{v \in N_{\tilde{S}}^w(t)} \mathbb{1}_{\{\tilde{S}_v < \tilde{T}_v\}} \middle| \mathcal{F}_T \right] \mathbb{1}_{\{T_w < S_w\}} \right]. \quad (7)$$

But by the strong Markov property and the many-to-one lemma,

$$\mathbb{P} \left[\sum_{v \in N_{\tilde{S}}^w(t)} \mathbb{1}_{\{\tilde{S}_v < \tilde{T}_v\}} \middle| \mathcal{F}_T \right] \mathbb{1}_{\{T_w < S_w\}} = \mathbb{Q}^{g^s, L^s} \left[\frac{e^S \zeta^{g^s, L^s}(0)}{\zeta^{g^s, L^s}(S^s)} \mathbb{1}_{\{S^s \leq T^s\}} \right] \Big|_{s=T_w} \mathbb{1}_{\{T_w < S_w\}}$$

where

$$g^s : \begin{cases} [0, t-s] & \rightarrow \mathbb{R} \\ u & \mapsto \sqrt{2}u - 1 - (b' \log t)(1-s/t) \end{cases}$$

and

$$L^s : \begin{cases} [0, t-s] & \rightarrow (0, \infty) \\ u & \mapsto L(s+u) = a(t+2-s-u)^{1/3}, \end{cases}$$

and we have defined two new stopping times

$$\mathcal{S}^s = \inf\{u \in (0, t-s] : \xi_u < g^s(u) + L^s(u) - 2\}$$

and

$$\mathcal{T}^s = \inf\{u \in [0, t-s] : \xi_u < g^s(u)\}.$$

Note that on the event $\{\xi_u - g^s(u) \in (0, L^s(u)) \ \forall u \leq t-s\}$,

$$\frac{\zeta^{g^s, L^s}(u)}{\zeta^{g^s, L^s}(0)} = e^{\sqrt{2}\xi_u - u + \int_s^{s+u} \frac{\pi^2}{2L(r)^2} dr + \frac{1}{2} \log L(s) - \frac{1}{2} \log L(s+u) + O(1)} \frac{\sin\left(\frac{\pi(\xi_u - g^s(u))}{L(s+u)}\right)}{\sin\left(\frac{-\pi g^s(0)}{L(s)}\right)}$$

so that

$$\frac{\zeta^{g^s, L^s}(\mathcal{S}^s)}{\zeta^{g^s, L^s}(0)} = e^{\mathcal{S}^s - (\sqrt{2}b' \log t)(1-s/t) + \sqrt{2}L(s) + \frac{3}{2} \log L(s) - \frac{3}{2} \log L(s+\mathcal{S}^s) + O(\log \log t)}.$$

Thus

$$\begin{aligned} & \mathbb{Q}^{g^s, L^s} \left[\frac{e^{\mathcal{S}^s} \zeta^{g^s, L^s}(0)}{\zeta^{g^s, L^s}(\mathcal{S}^s)} \mathbb{1}_{\{\mathcal{S}^s \leq \mathcal{T}^s\}} \right] \Big|_{s=T_w} \mathbb{1}_{\{T_w < S_w\}} \\ &= \mathbb{Q}^{g^s, L^s} \left[e^{(\sqrt{2}b' \log t)(1-s/t) - \sqrt{2}L(s) - \frac{3}{2} \log L(s) + \frac{3}{2} \log L(s+\mathcal{S}^s) + O(\log \log t)} \mathbb{1}_{\{\mathcal{S}^s \leq \mathcal{T}^s\}} \right] \Big|_{s=T_w} \mathbb{1}_{\{T_w < S_w\}} \\ &\leq e^{(\sqrt{2}b' \log t)(1-T_w/t) - \sqrt{2}L(T_w) + O(\log \log t)} \mathbb{1}_{\{T_w < S_w\}}. \end{aligned}$$

Plugging back into (7) we get

$$\mathbb{P}[\#\{v \in N_{\tilde{S}}(t) : T_v < S_v, \tilde{S}_v < \tilde{T}_v\}] \leq \mathbb{P} \left[\sum_{w \in N_T(t)} e^{(\sqrt{2}b' \log t)(1-T_w/t) - \sqrt{2}L(T_w) + O(\log \log t)} \mathbb{1}_{\{T_w < S_w\}} \right].$$

We then apply the many-to-one lemma again, and use the fact that

$$\xi_T = \mu T - L(0) + (b+b') \log t + 1$$

to get

$$\begin{aligned} & \mathbb{P}[\#\{v \in N_{\tilde{S}}(t) : T_v < S_v, \tilde{S}_v < \tilde{T}_v\}] \\ &\leq \mathbb{Q}^{f, L} \left[\frac{e^T \zeta^{f, L}(0)}{\zeta^{f, L}(T)} e^{(\sqrt{2}b' \log t)(1-T/t) - \sqrt{2}L(T) + O(\log \log t)} \mathbb{1}_{\{T < S\}} \right] \\ &= \mathbb{Q}^{f, L} \left[e^{T - \mu \xi_T + \frac{1}{2} \mu^2 T - \int_0^T \frac{\pi^2}{2L(u)^2} du + \frac{1}{2} \log L(T) - \frac{1}{2} \log L(0) + O(1)} \frac{\sin\left(\frac{-\pi f(0)}{L(0)}\right)}{\sin\left(\frac{\pi(\xi_T - f(T))}{L(T)}\right)} \right. \\ &\quad \left. \cdot e^{(\sqrt{2}b' \log t)(1-T/t) - \sqrt{2}L(T) + O(\log \log t)} \mathbb{1}_{\{T < S\}} \right] \\ &= \mathbb{Q}^{f, L} \left[e^{T - \frac{1}{2} \mu^2 T + \mu L(0) - \mu(b+b') \log t - \sqrt{2}L(0) + \sqrt{2}L(T) + \frac{3}{2} \log L(T) - \frac{3}{2} \log L(0)} \right. \\ &\quad \left. \cdot e^{(\sqrt{2}b' \log t)(1-T/t) - \sqrt{2}L(T) + O(\log \log t)} \mathbb{1}_{\{T < S\}} \right] \\ &= \mathbb{Q}^{f, L} [e^{-\sqrt{2}b \log t + \frac{3}{2} \log L(T) - \frac{3}{2} \log L(0) + O(\log \log t)} \mathbb{1}_{\{T < S\}}] \\ &\leq t^{-\sqrt{2}b} e^{O(\log \log t)}. \end{aligned}$$

□

As an easy corollary of the previous two lemmas, we obtain the upper bound for Theorem 2.

Proof of upper bound in Theorem 2. Note that

$$\mathbb{P}(\exists v \in N(t) : X_v(s) > g(s) \quad \forall s \leq t) \leq \mathbb{P}(\exists v : S_v \leq t) + \mathbb{P}(\exists v : T_v \leq t, \tilde{S}_v \leq t).$$

But Markov's inequality, combined with the bounds from Lemmas 10 and 11, shows that each of these two quantities is at most $t^{-\sqrt{2}b} e^{O(\log \log t)}$ as claimed. \square

Remark. If $b \geq 0$ then much of the work above is unnecessary. One can simply consider the $(g(s), L(s))$ -tube, without worrying about f , and use a more straightforward first moment method.

4.4 The lower bound in Theorem 2

For the lower bound the main problem is again when $b < 0$. We begin by proving the desired bound for $b \geq 0$, which is a fairly standard second moment method similar to that used in the proof of Theorem 1.

Proposition 12. For $a = a_c = 3^{1/3} \pi^{2/3} 2^{-1/2}$ and $b \geq 0$,

$$\mathbb{P}(\exists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t) \geq t^{-\sqrt{2}b-1/2} e^{O(\log \log t)}.$$

Proof. We define

$$f(u) = f_t(u) = \sqrt{2}u - a(t+1)^{1/3} + b \log t + 1$$

and

$$L(u) = L_t(u) = a(t+e-u)^{1/3}.$$

It clearly suffices to show that

$$\mathbb{P}(\exists v \in N(t) : X_v(u) - f(u) \in (0, L(u)) \quad \forall u \leq t) \geq t^{-\sqrt{2}b-1/2} e^{O(\log \log t)}. \quad (8)$$

As in Section 3 we define

$$\tilde{N}(s) = \#\{v \in N(s) : X_v(u) - f(u) \in (0, L(u)) \quad \forall u \leq s, \quad X_v(s) - f(s) \in (L(s) - 2, L(s) - 1)\}$$

and let

$$A_s^{(i)} = \{\xi_u^{(i)} - f(u) \in (0, L(u)) \quad \forall u \leq s\}$$

and

$$C_s^{(i)} = \{\xi_s^{(i)} - f(s) \in (L(s) - 2, L(s) - 1)\}$$

where $(\xi_u^{(1)}, u \geq 0)$ and $(\xi_u^{(2)}, u \geq 0)$ are standard Brownian motions that are equal until an independent exponentially distributed time T of parameter 2 and which move independently (given T and their common position at time T) after time T (see the proof of Theorem 1, or [5], for more details). Note that

$$\begin{aligned} \frac{1}{2} \int_0^s f'(u)^2 du &= s, \\ \int_0^s \frac{\pi^2}{2L(u)^2} du &= \sqrt{2}L(0) - \sqrt{2}L(s) \end{aligned}$$

and

$$f'(0)f(0) = -\sqrt{2}L(0) + \sqrt{2}b \log t + O(1).$$

It is also easy to check that the conditions of Lemma 8 hold.

By the many-to-one lemma and Lemma 8,

$$\mathbb{P}[\tilde{N}(t)] = t^{-\sqrt{2}b-1/2} e^{O(\log \log t)}.$$

By the many-to-two lemma [5], recalling that we set $\Theta_s = A_s^{(1)} \cap A_s^{(2)} \cap C_s^{(1)} \cap C_s^{(2)}$,

$$\mathbb{P}[\tilde{N}(t)^2] = \mathbb{P}[\tilde{N}(t)] + 2 \int_0^t e^{2t-s} \mathbb{P}(\Theta_t | T = s) ds.$$

If $s \leq \bar{\rho}_L(t)$, then by Lemma 8,

$$\begin{aligned} & \mathbb{P}\left(\Theta_t \mid A_T^{(1)} \cap A_T^{(2)} \cap \{\xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j]\} \cap \{T = s\}\right) \\ & \leq \sup_{z \in (j, j+1]} \mathbb{P}(\xi_u - f(s+u) + f(s) + L(s) - z \in (0, L(s+u)) \quad \forall u \leq t-s, \\ & \quad \xi_{t-s} - f(t) + f(s) + L(s) - z \in (L(t) - 2, L(t) - 1))^2 \\ & \leq \sup_{z \in (j, j+1]} e^{-2(t-s) - 2\sqrt{2}L(s) + 2\sqrt{2}L(t) - 2\sqrt{2}(z-L(s)) - 2\sqrt{2}L(t) + \log L(t) - \log L(s)} \\ & \quad \cdot \sin^2\left(\frac{\pi z}{L(s)}\right) \left(\int_{1-2/L(t)}^{1-1/L(t)} \sin(\pi\nu) d\nu\right)^2 \\ & \lesssim (j+1)^2 e^{-2\sqrt{2}j - 2t + 2s} (t+e-s)^{-1}. \end{aligned}$$

For $s \geq \rho_L$,

$$\begin{aligned} & \mathbb{P}\left(A_T^{(1)} \cap \{\xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j]\} \cap \{T = s\}\right) \\ & = \mathbb{P}(\xi_u - f(u) \in (0, L(u)) \quad \forall u \leq s, \quad \xi_s - f(s) \in (L(s) - j - 1, L(s) - j]) \\ & \leq e^{-s + \sqrt{2}j - (\sqrt{2}b + 1/2) \log t + \frac{1}{6} \log(t+e-s) + O(\log \log t)} \int_{1-(j+1)/L(s)}^{1-j/L(s)} \sin(\pi\nu) d\nu \\ & \leq (j+1) e^{\sqrt{2}j - s - \sqrt{2}b - 1/2} (t+e-s)^{-1/2}; \end{aligned} \tag{9}$$

if $s \in (1, \rho_L)$, then

$$\begin{aligned} & \mathbb{P}\left(A_T^{(1)} \cap \{\xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j]\} \cap \{T = s\}\right) \\ & \leq e^{-s + \sqrt{2}j - \sqrt{2}b \log t + O(\log \log t)} \frac{1}{L(s)} \frac{j+1}{L(s)} \frac{L(s)^2}{s^{3/2}} \\ & \leq (j+1) e^{\sqrt{2}j - s + O(\log \log t)} t^{-\sqrt{2}b} s^{-3/2}; \end{aligned}$$

and if $s \in (0, 1]$ then

$$\mathbb{P}\left(A_T^{(1)} \cap \{\xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j]\} \cap \{T = s\}\right) \leq (j+1) e^{\sqrt{2}j - s + O(\log \log t)} t^{-\sqrt{2}b}.$$

Putting these estimates together, if $s \in (0, 1]$ then

$$\mathbb{P}(\Theta_t | T = s) \leq e^{-2t+s+O(\log \log t)} t^{-\sqrt{2}b-1};$$

if $s \in (1, \rho_L)$ then

$$\mathbb{P}(\Theta_t | T = s) \leq e^{-2t+s+O(\log \log t)} t^{-\sqrt{2}b-1} s^{-3/2};$$

and if $s \in [\rho_L, \bar{\rho}_L(t)]$ then

$$\mathbb{P}(\Theta_t | T = s) \leq e^{-2t+s+O(\log \log t)} t^{-\sqrt{2}b-1/2} (t+e-s)^{-3/2}.$$

Thus

$$\int_0^{\bar{\rho}_L(t)} e^{2t-s} \mathbb{P}(\Theta_t | T = s) ds \leq t^{-\sqrt{2}b-1/2} e^{O(\log \log t)}.$$

If $s > \bar{\rho}_L(t)$, then $L(s) \leq (t + e - \bar{\rho}_L(t))^{1/3} = O(1)$ and $t - \bar{\rho}_L(t) = O(1)$, so directly from our earlier estimate (9) plus the fact that

$$\mathbb{P}\left(\Theta_t \mid A_T^{(1)} \cap A_T^{(2)} \cap \{\xi_T^{(1)} - f(T) \in (L(T) - j - 1, L(T) - j]\} \cap \{T = s\}\right) \leq 1$$

we see that

$$\int_{\bar{\rho}_L(t)}^t \mathbb{P}(\Theta_t \mid T = s) ds \leq e^{-2t+s+O(\log \log t)} t^{-\sqrt{2}b-1/2}.$$

As a result

$$\int_0^t e^{2t-s} \mathbb{P}(\Theta_t \mid T = s) ds \leq t^{-\sqrt{2}b-1/2} e^{O(\log \log t)},$$

so

$$\mathbb{P}(\tilde{N}(t) \geq 1) \geq \frac{\mathbb{P}[\tilde{N}(t)]^2}{\mathbb{P}[\tilde{N}(t)^2]} \geq t^{-\sqrt{2}b-1/2} e^{O(\log \log t)}.$$

□

To extend to the case $b < 0$ we need the following fairly straightforward estimate.

Lemma 13. *There exist $\delta > 0$ and $\varepsilon > 0$ such that for any $\mu \in [0, \sqrt{2}]$,*

$$\mathbb{P}\left(\#\{v \in N(s) : X_u(s) > \mu s\} \geq \delta e^{(1-\mu^2/2)s} s^{-3/2}\right) \geq \frac{\varepsilon}{1 + e^{(\mu^2/2-1)s} s^{3/2}} \quad \forall s > 0.$$

Proof. Let

$$\Gamma_s = \#\{v \in N(s) : X_v(u) < \mu u + 2 \quad \forall u \leq s, X_v(s) \in (\mu s, \mu s + 1)\}$$

and

$$V_s = \frac{1}{2}(2 + \mu s - \xi_s) e^{\mu \xi_s - \frac{1}{2} \mu^2 s} \mathbb{1}_{\{\xi_u < \mu u + 2 \quad \forall u \leq s\}}$$

and define a new measure $\hat{\mathbb{Q}}$ by setting

$$\left. \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} \right|_{\mathcal{G}_s} = V_s.$$

Then it is well-known that under $\hat{\mathbb{Q}}$, $(2 + \mu s - \xi_s, s \geq 0)$ is a Bessel-3 process started from 2 (see for example [11]). By the many-to-one lemma, if $(Y_s, s \geq 0)$ is a Bessel-3 process under \mathbb{P} , then

$$\begin{aligned} \mathbb{P}[\Gamma_s] &= e^s \hat{\mathbb{Q}} \left[\frac{1}{V_s} \mathbb{1}_{\{\xi_s \in (\mu s, \mu s + 1)\}} \right] \\ &\asymp e^{(1-\mu^2/2)s} \hat{\mathbb{Q}}(\xi_s \in (\mu s, \mu s + 1)) \\ &= e^{(1-\mu^2/2)s} \mathbb{P}(Y_s \in (1, 2)) \\ &\asymp s^{-3/2} e^{(1-\mu^2/2)s}. \end{aligned}$$

By the many-to-few lemma [5], with $V_s^{(i)}$ and $Y_s^{(i)}$ defined as usual with $\xi_s^{(i)}$ in place of ξ_s ,

$$\begin{aligned} \mathbb{P}[\Gamma_s^2] &= \mathbb{P}[\Gamma_s] + e^{2s} \hat{\mathbb{Q}} \left[e^T \frac{V_T^{(1)}}{V_s^{(1)} V_s^{(2)}} \mathbb{1}_{\{T \leq s, \xi_s^{(1)} \in (\mu s, \mu s + 1), \xi_s^{(2)} \in (\mu s, \mu s + 1)\}} \right] \\ &\asymp \mathbb{P}[\Gamma_s] + e^{(2-\mu^2)s} \hat{\mathbb{Q}} \left[(2 + \mu T - \xi_T^{(1)}) e^{T + \mu \xi_T^{(1)} - \mu^2 T/2} \mathbb{1}_{\{T \leq s, \xi_s^{(1)} \in (\mu s, \mu s + 1), \xi_s^{(2)} \in (\mu s, \mu s + 1)\}} \right] \\ &= \mathbb{P}[\Gamma_s] + e^{(2-\mu^2)s} \mathbb{P} \left[Y_T^{(1)} e^{(1+\mu^2/2)T + \mu Y_T^{(1)}} \mathbb{1}_{\{T \leq s, Y_s^{(1)} \in (1, 2), Y_s^{(2)} \in (1, 2)\}} \right]. \end{aligned}$$

But by Lemma 4 of [12], provided $\mu \leq \sqrt{2}$,

$$\mathbb{P} \left[Y_T^{(1)} e^{(1+\mu^2/2)T + \mu Y_T^{(1)}} \mathbb{1}_{\{T \leq s, Y_s^{(1)} \in (1, 2), Y_s^{(2)} \in (1, 2)\}} \right] \lesssim s^{-3}.$$

Thus

$$\mathbb{P}[\Gamma_s^2] \lesssim e^{(1-\mu^2/2)s} s^{-3/2} + e^{(2-\mu^2)s} s^{-3}$$

and hence, by the Paley-Zygmund inequality,

$$\mathbb{P}(\Gamma_s \geq \mathbb{P}[\Gamma_s]/2) \geq \frac{\mathbb{P}[\Gamma_s]^2}{4\mathbb{P}[\Gamma_s^2]} \gtrsim \frac{e^{(2-\mu^2)s} s^{-3}}{e^{(1-\mu^2/2)s} s^{-3/2} + e^{(2-\mu^2)s} s^{-3}} = \frac{1}{1 + e^{(\mu^2/2-1)s} s^{3/2}}$$

which completes the proof. \square

We now combine Proposition 12 and Lemma 13 to obtain our lower bound for all $b \in \mathbb{R}$.

Proof of lower bound in Theorem 2. By Proposition 12 it remains to consider $b \in [-1/2\sqrt{2}, 0)$. We begin by applying Lemma 13 at time $-b \log^2 t / \log \log t$ with $\mu = \sqrt{2} - \log \log t / \log t$.

Choose $C > -b/2 + 3$ and $c < -b/2 + 3$ and let

$$H_t = \left\{ v \in N \left(-b \frac{\log^2 t}{\log \log t} \right) : X_v \left(-b \frac{\log^2 t}{\log \log t} \right) > -\sqrt{2}b \frac{\log^2 t}{\log \log t} + b \log t \right\}$$

and

$$J_t = \{ \#H_t \geq t^{-\sqrt{2}b} e^{-C \log \log t} \}.$$

Since

$$\left(\sqrt{2} - \frac{\log \log t}{\log t} \right) \left(-b \frac{\log^2 t}{\log \log t} \right) = -\sqrt{2}b \frac{\log^2 t}{\log \log t} + b \log t$$

and (for large t and any $\delta > 0$)

$$t^{-\sqrt{2}b} e^{-C \log \log t} \leq \delta e^{\left(1 - \frac{1}{2} \left(\sqrt{2} - \frac{\log \log t}{\log t} \right)^2\right) \left(-b \frac{\log^2 t}{\log \log t} \right)} \left(-b \frac{\log^2 t}{\log \log t} \right)^{-3/2} \leq t^{-\sqrt{2}b} e^{-c \log \log t},$$

we see by Lemma 13 that there exists $\varepsilon > 0$ such that

$$\mathbb{P}(J_t) \geq \varepsilon$$

for all large t . Also, for large t , by the many-to-one lemma,

$$\begin{aligned} & \mathbb{P} \left(\exists v \in N \left(-b \frac{\log^2 t}{\log \log t} \right), \quad u \leq -b \frac{\log^2 t}{\log \log t} : X_v(u) \leq \sqrt{2}u - at^{1/3} + b \log t \right) \\ & \leq e^{-b \frac{\log^2 t}{\log \log t}} \mathbb{P} \left(\exists u \leq -b \frac{\log^2 t}{\log \log t} : \xi_u \leq -\frac{a}{2} t^{1/3} \right) \end{aligned}$$

which converges to 0 as $t \rightarrow \infty$. Thus, if we set

$$\tilde{H}_t = \left\{ v \in H_t : X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq -b \frac{\log^2 t}{\log \log t} \right\}$$

and

$$\tilde{J}_t = \{ \#\tilde{H}_t \geq t^{-\sqrt{2}b} e^{-C \log \log t} \}$$

then for large t

$$\mathbb{P}(\tilde{J}_t) \geq \varepsilon/2.$$

Now,

$$\begin{aligned} & \mathbb{P}(\nexists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t | \tilde{J}_t) \\ & \leq \mathbb{P} \left(\bigcap_{w \in \tilde{H}_t} \left\{ \nexists v \in N(t) : w \leq v, \quad X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t \right\} \middle| \tilde{J}_t \right) \\ & = \mathbb{P} \left[\prod_{w \in \tilde{H}_t} \mathbb{P} \left(\nexists v \in N(t) : w \leq v, \quad X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t \middle| \mathcal{F}_{-b \frac{\log^2 t}{\log \log t}} \right) \middle| \tilde{J}_t \right]. \end{aligned}$$

But if $w \in \tilde{H}_t$, then

$$\begin{aligned}
& \mathbb{P} \left(\exists v \in N(t) : w \leq v, \quad X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t \mid \mathcal{F}_{-b \frac{\log^2 t}{\log \log t}} \right) \\
& \leq \mathbb{P} \left(\exists v \in N \left(t + b \frac{\log^2 t}{\log \log t} \right) : X_v(u) > \sqrt{2}u - at^{1/3} \quad \forall u \leq t + b \frac{\log^2 t}{\log \log t} \right) \\
& \leq \mathbb{P}(\exists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} \quad \forall u \leq t) \\
& \leq 1 - t^{-1/2} e^{-C' \log \log t}
\end{aligned}$$

for some $C' \geq 0$, where the last line follows from Proposition 12 with $b = 0$. Thus

$$\begin{aligned}
& \mathbb{P}(\exists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t \mid \tilde{J}_t) \\
& \leq \left(1 - t^{-1/2} e^{-C' \log \log t} \right)^{t^{-\sqrt{2}b} \exp(-C \log \log t)} \\
& = 1 - t^{-\sqrt{2}b-1/2} e^{O(\log \log t)}
\end{aligned}$$

and so

$$\mathbb{P}(\exists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t \mid \tilde{J}_t) \geq t^{-\sqrt{2}b-1/2} e^{O(\log \log t)}.$$

Finally,

$$\begin{aligned}
& \mathbb{P}(\exists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t) \\
& \geq \mathbb{P}(\{\exists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \quad \forall u \leq t\} \cap \tilde{J}_t) \\
& \geq \frac{\varepsilon}{2} t^{-\sqrt{2}b-1/2} e^{O(\log \log t)}
\end{aligned}$$

as required. \square

5 Proof of Theorem 3

We recall the setup of Theorem 3. For $v \in N(t)$, we let $\lambda(v, t) = \sup_{s \in [0, t]} \{\sqrt{2}s - X_v(s)\}$ and define $\Lambda(t) = \min_{v \in N(t)} \lambda(v, t)$. Then we wish to show that

$$\liminf_{t \rightarrow \infty} \frac{\Lambda(t) - at^{1/3}}{\log t} \in \left[-\frac{1}{2\sqrt{2}}, 0 \right]$$

almost surely, and

$$\limsup_{t \rightarrow \infty} \frac{\Lambda(t) - at^{1/3}}{\log t} \in \left[\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$$

almost surely, where $a = a_c = 3^{1/3} \pi^{2/3} 2^{-1/2}$.

The theorem states the following: at any large time t , we can find particles that have stayed above $\sqrt{2}u - at^{1/3} + b \log t$ for all times $u \leq t$, as long as $b < -1/2\sqrt{2}$. However, for any $b > 0$, we can find an arbitrarily large t such that no particle has stayed above $\sqrt{2}u - at^{1/3} + b \log t$ for all times $u \leq t$. On the other hand, for any $b < 1/3\sqrt{2}$, we can also find an arbitrarily large t such that there are particles that have stayed above $\sqrt{2}u - at^{1/3} + b \log t$ for all times $u \leq t$. Finally, for any $b > 1/\sqrt{2}$, at any large time t , there do not exist particles that stay above $\sqrt{2}u - at^{1/3} + b \log t$ for all times $u \leq t$. We prove four lemmas, each of which represents one of these four statements.

Lemma 14.

$$\liminf_{t \rightarrow \infty} \frac{\Lambda(t) - at^{1/3}}{\log t} \leq 0 \quad \text{almost surely.}$$

Proof. To rephrase the statement of the lemma, we show that for any $\varepsilon > 0$ there are arbitrarily large times such that no particles have stayed above $\sqrt{2}u - at^{1/3} + \varepsilon \log t$ for all $s \leq t$. Choose $\delta < \varepsilon/2$, let $t_1 = 1$ and for $n > 1$ let $t_n = \exp(\frac{1}{\delta} \exp(2t_{n-1}))$. Define

$$E_n = \{\exists v \in N(t_n) : X_v(u) > \sqrt{2}u - at_n^{1/3} + \varepsilon \log t_n \quad \forall u \leq t_n\}$$

and

$$F_n = \{|N(t_n)| \leq e^{2t_n}, \quad |X_v(t_n)| \leq \sqrt{2}t_n \quad \forall v \in N(t_n)\}.$$

We know that F_n occurs for all large n , so it suffices to show that

$$\mathbb{P}\left(\bigcap_{k \geq n} (E_k \cap F_k)\right) = \lim_{N \rightarrow \infty} \prod_{k=n}^N \mathbb{P}\left(E_k \cap F_k \middle| \bigcap_{j=n}^{k-1} (E_j \cap F_j)\right) = 0 \text{ for all } n \geq 0.$$

For a particle v , let $N^v(t)$ be the set of descendants of v at time t , and let E_n^v be the event that some descendant of v at time t_n has stayed above $\sqrt{2}u - at_n^{1/3} - \varepsilon \log t_n$ for all times $u \leq t_n$. Also let $s_n = t_n - t_{n-1}$. Then if $v \in N(t_{n-1})$ and $X_v(t_{n-1}) \leq \sqrt{2}t_{n-1}$,

$$\begin{aligned} & \mathbb{P}(E_n^v | \mathcal{F}_{t_{n-1}}) \\ &= \mathbb{P}(\exists w \in N^v(t_n) : X_w(u) > \sqrt{2}u - at_n^{1/3} + \varepsilon \log t_n \quad \forall u \leq t_n | \mathcal{F}_{t_{n-1}}) \\ &= \mathbb{P}(\exists w \in N(s_n) : X_w(u) > \sqrt{2}u - at_n^{1/3} + \varepsilon \log t_n \quad \forall u \leq s_n) \\ &\leq \mathbb{P}\left(\exists w \in N(s_n) : X_w(u) > \sqrt{2}u - as_n^{1/3} + \frac{\varepsilon}{2} \log s_n \quad \forall u \leq s_n\right). \end{aligned}$$

Noting that $s_n \geq t_n/2$, by the upper bound in Theorem 2 we get

$$\mathbb{P}(E_n^v | \mathcal{F}_{t_{n-1}}) \mathbb{1}_{\{|X_v(t_{n-1})| \leq e^{2t_{n-1}}\}} \leq s_n^{-\varepsilon/\sqrt{2}} e^{O(\log \log s_n)} \leq t_n^{-\varepsilon/\sqrt{2}} e^{O(\log \log t_n)}.$$

Thus, since $e^{2t_{k-1}} = \delta \log t_k$,

$$\begin{aligned} \mathbb{P}\left(E_k \cap F_k \middle| \bigcap_{j=n}^{k-1} (E_j \cap F_j)\right) &\leq \mathbb{P}\left(E_k \middle| \bigcap_{j=n}^{k-1} (E_j \cap F_j)\right) \\ &\leq \mathbb{P}\left(\bigcup_{v \in N(t_{k-1})} E_k^v \middle| \bigcap_{j=n}^{k-1} (E_j \cap F_j)\right) \\ &\leq e^{2t_{k-1}} t_k^{-\varepsilon/\sqrt{2}} e^{O(\log \log t_k)} \\ &\leq t_k^{-\varepsilon/\sqrt{2} + \sqrt{2}\delta} e^{O(\log \log t_k)}. \end{aligned}$$

Since we chose $\delta < \varepsilon/2$, this tends to zero as $k \rightarrow \infty$. \square

Lemma 15.

$$\limsup_{t \rightarrow \infty} \frac{\Lambda(t) - at^{1/3}}{\log t} \leq 1/\sqrt{2} \quad \text{almost surely.}$$

Proof. We show that for large t and any $\varepsilon > 0$, there are no particles that stay above $\sqrt{2}u - at^{1/3} + (1/\sqrt{2} + 2\varepsilon) \log t$ for all times $u \leq t$. By the upper bound in Theorem 2,

$$\begin{aligned} \mathbb{P}(\exists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} + (1/\sqrt{2} + \varepsilon) \log t \quad \forall u \leq t) \\ \leq t^{-\sqrt{2}(1/\sqrt{2} + \varepsilon)} e^{O(\log \log t)} = t^{-1 - \sqrt{2}\varepsilon} e^{O(\log \log t)}. \end{aligned}$$

Thus for any lattice times $t_n \rightarrow \infty$ (that is, times t_n such that there exists $\delta > 0$ with $t_{n+1} - t_n = \delta$ for all n), by Borel-Cantelli

$$\mathbb{P}(\exists v \in N(t_n) : X_v(u) > \sqrt{2}u - at_n^{1/3} + (1/\sqrt{2} + \varepsilon) \log t_n \quad \forall u \leq t_n \text{ for infinitely many } n) = 0.$$

By choosing $\delta = t_{n+1} - t_n$ small enough, we may ensure that for large n if $t \in (t_n, t_{n+1})$ then $-at^{1/3} + (1/\sqrt{2} + 2\varepsilon) \log t > -at_n^{1/3} + (1/\sqrt{2} + \varepsilon) \log t_n$, which completes the proof. \square

Lemma 16.

$$\liminf_{t \rightarrow \infty} \frac{\Lambda(t) - at^{1/3}}{\log t} \geq -\frac{1}{2\sqrt{2}} \quad \text{almost surely.}$$

Proof. We show that for large t and any $\varepsilon > 0$, there are always particles that have stayed above $\sqrt{2}u - at^{1/3} - (1/2\sqrt{2} + 2\varepsilon)\log t$ for all times $u \leq t$. Let

$$A_t = \{\exists v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} - (1/2\sqrt{2} + \varepsilon)\log t \quad \forall u \leq t\}$$

and

$$B_t = \{|N(\varepsilon \log t)| \geq t^{\varepsilon/2}, |X_v(\varepsilon \log t)| \leq \sqrt{2}\varepsilon \log t \quad \forall v \in N(\varepsilon \log t)\}.$$

As before we write $N^v(t)$ for the set of descendants of particle v that are alive at time t . Let $l_t = t - \varepsilon \log t$. Then for all large t ,

$$\begin{aligned} & \mathbb{P}(A_t \cap B_t) \\ & \leq \mathbb{P} \left[\prod_{v \in N(\varepsilon \log t)} \mathbb{P}(\exists w \in N^v(t) : X_w(u) > \sqrt{2}u - at^{1/3} - (1/\sqrt{2} + \varepsilon)\log t \quad \forall u \leq t | \mathcal{F}_{\varepsilon \log t}) \mathbb{1}_{B_t} \right] \\ & \leq \mathbb{P} \left[\prod_{v \in N(\log t)} \mathbb{P}(\exists w \in N(l_t) : X_w(u) > \sqrt{2}u - at^{1/3} - (1/\sqrt{2} + \varepsilon)\log t \quad \forall u \leq l_t) \mathbb{1}_{B_t} \right] \\ & \leq \mathbb{P} \left(\exists w \in N(l_t) : X_w(u) > \sqrt{2}u - al_t^{1/3} - (1/\sqrt{2} + \varepsilon)\log l_t \quad \forall u \leq l_t \right)^{t^{\varepsilon/2}}. \end{aligned}$$

By the lower bound in Theorem 2 there exists γ such that this is at most

$$(1 - e^{-\gamma \log \log t})^{t^{\varepsilon/2}}.$$

Thus by Borel-Cantelli, for any lattice times $t_n \rightarrow \infty$, $\mathbb{P}(A_{t_n} \cap B_{t_n} \text{ infinitely often}) = 0$. But for all large t , $|N(\varepsilon \log t)| \geq e^{\frac{\varepsilon}{2} \log t} = t^{\varepsilon/2}$ and $X_v(\varepsilon \log t) \geq -\sqrt{2}\varepsilon \log t$ for all $v \in N(\varepsilon \log t)$, so we deduce that $\mathbb{P}(A_{t_n} \text{ infinitely often}) = 0$. Then if we choose $t_n - t_{n-1}$ small enough, $-at^{1/3} - (1/2\sqrt{2} + 2\varepsilon)\log t < -at_n^{1/3} - (1/2\sqrt{2} + \varepsilon)\log t_n$ for all $t \in (t_{n-1}, t_n)$, so the result holds. \square

Lemma 17.

$$\limsup_{t \rightarrow \infty} \frac{\Lambda(t) - at^{1/3}}{\log t} \geq 1/3\sqrt{2} \quad \text{almost surely.}$$

Proof. Rather than a fixed $t > 0$, for this proof we will need to consider two different times s and t . We thus emphasise the t -dependence of our functions f and L , and introduce a new function J . For $b \geq 0$ and $a = a_c$, let

$$f_t(u) = \sqrt{2}u - at^{1/3} + b \log t + 1,$$

$$L_t(u) = a(t + e - u)^{1/3}$$

and

$$J_t(u) = a(t + e - u)^{1/3} - b \log t.$$

We will be interested in the set

$$V_t(u) = \{v \in N(u) : X_v(r) - f_t(r) \in (0, J_t(r)) \quad \forall r \leq u \wedge t / \log t, X_v(r) - f_t(r) \in (0, L_t(r)) \quad \forall r \in [0, u]\}.$$

We shall show that when we choose $b \leq 1/2\sqrt{2}$, there exist arbitrarily large times at which $V_t(t)$ is non-empty. To this end define

$$I_n^b = \int_n^{2n} \mathbb{1}_{\{V_t(t) \neq \emptyset\}} dt.$$

By using Lemma 8 together with the fact that $\int_0^{t/\log t} 1/J_t(u)^2 du = \int_0^{t/\log t} 1/L_t(u)^2 du + O(1)$, we see that

$$\mathbb{P}[V_t(t)] \asymp t^{-1/2-\sqrt{2}b}.$$

Since

$$V_t(t) \subseteq \{v \in N(t) : X_v(r) - f_t(r) \in (0, L_t(r)) \ \forall r \in [0, t]\},$$

by the second moment calculation in Proposition 12 together with Cauchy-Schwarz we see that

$$\mathbb{P}(V_t(t) \neq \emptyset) \geq t^{-1/2-\sqrt{2}b} e^{O(\log \log t)}.$$

Thus

$$\mathbb{P}[I_n^b] = \int_n^{2n} \mathbb{P}(V_t(t) \neq \emptyset) dt \geq \int_n^{2n} e^{O(\log \log t)} t^{-\sqrt{2}b-1/2} dt = n^{1/2-\sqrt{2}b} e^{O(\log \log n)}.$$

We now move on to an upper bound for the second moment of I_n , noting that

$$\begin{aligned} \mathbb{P}[(I_n^b)^2] &= \mathbb{P} \left[\int_n^{2n} \int_n^{2n} \mathbb{1}_{\{V_s(s) \neq \emptyset, V_t(t) \neq \emptyset\}} ds dt \right] \\ &= 2 \int_n^{2n} \int_n^t \mathbb{P}(V_s(s) \neq \emptyset, V_t(t) \neq \emptyset) ds dt. \end{aligned}$$

Now, for $s \leq t$,

$$\mathbb{P}(V_s(s) \neq \emptyset, V_t(t) \neq \emptyset) \leq \mathbb{P}[\#V_s(s) \cdot \#V_t(t)] = \mathbb{P}[\#V_s(s) \cdot \mathbb{P}[\#V_t(t) | \mathcal{F}_s]].$$

Applying the many-to-few lemma [5],

$$\begin{aligned} \mathbb{P}(V_s(s) \neq \emptyset, V_t(t) \neq \emptyset) &\leq e^t \mathbb{P}(\xi_s \in V_s(s), \xi_t \in V_t(t)) + 2 \int_0^s e^{t+s-u} \mathbb{P}(\xi_s^{(1)} \in V_s(s), \xi_t^{(2)} \in V_t(t) | T = u) du \end{aligned}$$

where $\xi_r \in N_r(r)$ represents the obvious abuse of notation.

The calculations now required are very similar to those seen in the proofs of the lower bounds in Theorems 1 and 2. We concentrate on estimating the second term on the right-hand side above, in the case when u, s and t are all well separated. The other cases are not very different, and (again just as in Theorems 1 and 2) involve applying the latter part of Lemma 8.

First, if $u > \rho_{L_s}$, then applying Lemma 8,

$$\begin{aligned} \mathbb{P}(\xi_u \in V_s(u), \xi_u - f_s(u) \in (j, j+1]) &\leq e^{-u-\sqrt{2}L_s(0)+\sqrt{2}L_s(u)+\sqrt{2}L_s(0)-\sqrt{2}b \log s - \sqrt{2}j + \frac{1}{2} \log L_s(u) - \frac{3}{2} \log L_s(0) + O(\log \log s)} \\ &\quad \cdot \int_{j/L_s(u)}^{(j+1)/L_s(u)} \sin(\pi \nu) d\nu. \end{aligned}$$

Second, if $u < \bar{\rho}_{L_s}(s)$, then

$$\begin{aligned} \mathbb{P}(\xi_s^{(1)} \in V_s(s) | \xi_u^{(1)} - f_s(u) \in (j, j+1], \xi_u^{(1)} \in V_s(u)) &\leq \sup_{z \in [j, j+1]} \mathbb{P}(\xi_r - \sqrt{2}r + z \in (0, L_s(u+r)) \ \forall r \leq s-u) \\ &\lesssim e^{-(s-u)-\sqrt{2}L_s(u)+\sqrt{2}j-\frac{1}{2} \log L_s(u)} \sin\left(\frac{-\pi(j+1)}{L_s(u)}\right). \end{aligned}$$

Third, if $u < \bar{\rho}_{L_t}(s)$, then

$$\begin{aligned} & \mathbb{P}(\xi_t^{(2)} \in V_t(t) | \xi_u^{(2)} - f_s(u) \in (j, j+1], \xi_u^{(2)} \in V_s(u)) \\ & \leq \sup_{z \in [j, j+1]} \mathbb{P}(\xi_r - f_t(u+r) + f_s(u) + z \in (0, L_t(u+r)) \quad \forall r \leq t-u) \\ & \lesssim e^{-(t-u) - \sqrt{2}L_t(u) - \sqrt{2}(f_t(u) - f_s(u) - j) - \frac{1}{2} \log L_t(u)} \sin \left(\frac{-\pi(f_t(u) - f_s(u) - j)}{L_t(u)} \right). \end{aligned}$$

Putting these three estimates together, we see that if $u \in (\rho_{L_s}, s/\log s]$,

$$\begin{aligned} & e^{t+s-u} \mathbb{P}(\xi_s^{(1)} \in V_s(s), \xi_t^{(2)} \in V_t(t) | T = u) \\ & \leq e^{\sqrt{2}J_s(u) - \sqrt{2}L_t(u) + \sqrt{2}L_t(0) - \sqrt{2}L_s(0) - \sqrt{2}b \log t - 3 \log L_s(u) - \frac{3}{2} \log L_s(0) - \frac{1}{2} \log L_t(u) + O(\log \log t)} \\ & \quad \cdot \sin \left(\frac{-\pi(f_t(u) - f_s(u) - j)}{L_t(u)} \right). \end{aligned}$$

Noting that

$$e^{\sqrt{2}L_s(u) - \sqrt{2}L_t(u) + \sqrt{2}L_t(0) - \sqrt{2}L_s(0)} \sin \left(\frac{-\pi(f_t(u) - f_s(u) - j)}{L_t(u)} \right) \lesssim \frac{1}{L_t(u)},$$

we get

$$e^{t+s-u} \mathbb{P}(\xi_s^{(1)} \in V_s(s), \xi_t^{(2)} \in V_t(t) | T = u) \leq e^{O(\log \log t)} s^{-\sqrt{2}b-1/2} t^{-\sqrt{2}b} (s-u)^{-1} (t-u)^{-1/2}. \quad (10)$$

Thus if $b \geq -1/2$ then

$$\int_n^{2n} \int_n^t \int_{\rho_{L_s}}^{s/\log s} e^{t+s-u} \mathbb{P}(\xi_s^{(1)} \in V_s(s), \xi_t^{(2)} \in V_t(t) | T = u) du \leq n^{1-2\sqrt{2}b} e^{O(\log \log n)}.$$

If $u \geq s/\log s$, we note that

$$L_s(u) - L_t(u) + L_t(0) - L_s(0) \lesssim \frac{1}{3} t^{-2/3} \frac{s}{\log s} - \frac{1}{3} s^{-2/3} \frac{s}{\log s} \leq \frac{1}{9} \frac{t^{1/3}}{\log t} \left(\frac{s}{t} - 1 \right)$$

so

$$e^{t+s-u} \mathbb{P}(\xi_s^{(1)} \in V_s(s), \xi_t^{(2)} \in V_t(t) | T = u) \leq e^{O(\log \log t) + \frac{\sqrt{2}}{9} \frac{t^{1/3}}{\log t} \left(\frac{s}{t} - 1 \right)} s^{-1/2} t^{-\sqrt{2}b} (s-u)^{-1} (t-u)^{-1/6}.$$

Thus if $b \geq -1/2$ then

$$\begin{aligned} & \int_n^{2n} \int_n^{t-\Gamma t^{2/3} \log^2 t} \int_{s/\log s}^{\bar{\rho}_{L_s}(s)} e^{t+s-u} \mathbb{P}(\xi_s^{(1)} \in V_s(s), \xi_t^{(2)} \in V_t(t) | T = u) du \\ & \leq n^{7/6-2\sqrt{2}b-\Gamma\sqrt{2}/9} e^{O(\log \log n)}; \end{aligned}$$

choosing $\Gamma = 6/\sqrt{2}$ we obtain

$$\int_n^{2n} \int_n^{t-\Gamma t^{2/3} \log^2 t} \int_{s/\log s}^{\bar{\rho}_{L_s}(s)} e^{t+s-u} \mathbb{P}(\xi_s^{(1)} \in V_s(s), \xi_t^{(2)} \in V_t(t) | T = u) du \leq n^{1/2-\sqrt{2}b} e^{O(\log \log n)}.$$

Also, again from (10),

$$\int_n^{2n} \int_{t-\Gamma t^{2/3} \log^2 t}^t \int_{s/\log s}^{s/2} e^{t+s-u} \mathbb{P}(\xi_s^{(1)} \in V_s(s), \xi_t^{(2)} \in V_t(t) | T = u) du \leq n^{-\sqrt{2}b+2/3}.$$

Combining these with other similar estimates, and then integrating over u , s , and t , we obtain that (for $b \geq 0$)

$$\mathbb{P}[(I_n^b)^2] \leq (n^{\frac{2}{3}-\sqrt{2}b} + n^{1-2\sqrt{2}b})e^{O(\log \log n)}.$$

By Cauchy-Schwarz,

$$\mathbb{P}(I_n^b > 0) \geq \frac{\mathbb{P}[I_n^b]^2}{\mathbb{P}[(I_n^b)^2]} \geq \frac{n^{1-2\sqrt{2}b}}{n^{2/3-\sqrt{2}b} + n^{1-2\sqrt{2}b}}e^{O(\log \log n)} = \frac{e^{O(\log \log n)}}{n^{\sqrt{2}b-1/3} + 1}.$$

Finally, let $b < 1/3\sqrt{2}$, choose $\varepsilon < \frac{1/3\sqrt{2}-b}{2}$ and define

$$\hat{N}(t) = \{v \in N(t) : |X_v(u)| \leq 2t \ \forall u \leq t\},$$

$$\mathcal{A}_n = \{\#\hat{N}(\varepsilon \log n) \geq n^{\varepsilon/2}\}$$

and

$$\mathcal{B}_n = \{\exists t \in [\varepsilon \log n + n, \varepsilon \log n + 2n], v \in N(t) : X_v(u) > \sqrt{2}u - at^{1/3} + b \log t \ \forall u \leq t\}.$$

When $t \in [\varepsilon \log n + n, \varepsilon \log n + 2n]$, any particle v that ventures below $\sqrt{2}u - a(t + \alpha_t)^{1/3} + b \log t$ at some time $u \leq t$ must either fall below $-2\varepsilon \log n$ at some time $u \leq \varepsilon \log n$, or there must exist a time $u \in [\varepsilon \log n, t]$ such that $X_v(u) - X_v(\varepsilon \log n) \leq \sqrt{2}u - a(t + \alpha_t)^{1/3} + (b + 2\varepsilon) \log t$. Thus, by conditioning on $\mathcal{F}_{\varepsilon \log n}$,

$$\begin{aligned} \mathbb{P}(\mathcal{A}_n, \mathcal{B}_n^c) &\leq \mathbb{P} \left[\mathbb{1}_{\mathcal{A}_n} \prod_{v \in \hat{N}(\varepsilon \log n)} (1 - \mathbb{P}(I_n^{b+2\varepsilon} > 0)) \right] \\ &\leq (1 - e^{O(\log \log n)} n^{\varepsilon/2}). \end{aligned}$$

By Borel-Cantelli, the probability that both \mathcal{A}_n and \mathcal{B}_n^c occur infinitely often is zero. But we know from standard estimates that \mathcal{A}_n occurs for all large n , so \mathcal{B}_n must occur for all large n . This completes the proof. \square

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